Tau Function Approach to Theta Functions

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Abstract

We study theta functions of a Riemann surface of genus g from the view point of τ-function of a hierarchy of soliton equations. We study two kinds of series expansions. One is the Taylor expansion at any point of the theta divisor. We describe the initial term of the expansion by the Schur function corresponding to the partition determined by the gap sequence of a certain flat line bundle. The other is the expansion of the theta function and its certain derivatives in one of the variables on the Abel-Jacobi images of k points on a Riemann surface with $k \leq g$. We determine the initial term of the expansion as certain derivatives of the theta function successively. As byproducts, firstly we obtain a refinement of Riemann's singularity theorem. Secondly we determine normalization constants of higher genus sigma functions of a Riemann surface, defined by Korotkin and Shramchenko, such that they become modular invariant.

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1 Introduction

The notion of τ-function of an integrable hierarchy of soliton equations is considered as an extension of the notion of theta function. This is not only because the τ-function becomes a theta function in a special case but also because the structural similarities like addition formulae [34] which are equivalent to the the hierarchy itself [36, 30]. More concretely let us consider the KP-hierarchy which is considered to be a universal system of integrable differential equations in the sense that various soliton equations are derived as special cases of it. Its solution space is determined by Sato [34, 33] as a certain infinite dimensional Grassmann manifold called the universal Grassmann manifold (UGM). In other words there is a one to one correspondence between solutions of the KP-hierarchy(τ-functions) and points of UGM.

Sato's theory on the KP-hierarchy was successfully applied to the study of Novikov's conjecture [28, 23, 31]. The results show that theta functions of algebraic curves are characterized by the KP-hierarchy among those of principally polarized Abelian varieties (ppAv). It means that tau functions of the KP-hierarchy supplied by the general properties of theta functions of ppAv can produce all the properties of theta functions corresponding to Riemann surfaces. Therefore it is quite natural to study theta functions by way of τ-functions.

However it seems that a τ-function approach in the study of theta functions is not fully developed yet. In papers [25, 27, 9, 2] we have studied the higher genus sigma functions of various algebraic curves from the view point of τ-functions. The aim of this paper is to develop these researches further and to add examples which are effectively studied by an approach of τ-functions.

We study two kinds of series expansions of the theta function of a Riemann surface. One is the series expansion at any point on the theta divisor. We determine the initial term, with respect to certain weight, of the expansion as the Schur function with the partition determined from the gap sequence of the flat line bundle corresponding to a point on the theta divisor. The other is the expansion of the theta function and certain derivatives of the theta function with respect to one of the variables on the Abel-Jacobi images of k points on a Riemann surface of genus g with $k \leq g$. We determine the initial term of the expansion as a certain explicit derivative of the theta function.

As a consequence of the study on the expansions we get an extension and a refinement of Riemann's singularity theorem. As another corollary we determine normalization constants of higher genus sigma functions of a Riemann surface introduced by Korotkin and Shramchenko [18] so that they are modular invariant. In their paper the modular invariance of sigma functions is proved up to multiplications of certain roots of unity. We propose apparently different normalization constants from theirs and prove the modular invariance. Let us explain our results in more detail.

Let X be a compact Riemann surface of genus g, $\{\alpha_i, \beta_i\}$ a canonical homology basis, Ω the normalized period matrix and $\theta(Z|\Omega)$ Riemann's theta function. We consider the data $(X, \{\alpha_i, \beta_i\}, p_\infty, e)$ consisting of X, $\{\alpha_i, \beta_i\}$ as above, a point p_∞ on X and a point e of the theta divisor. We associate a partition to such a data as follows.

To this end we need a notion of gaps of a line bundle. Let L be a holomorphic line

bundle on X of degree zero, which we call a flat line bundle. A non-negative integer n is called a gap of L at p_∞ if there does not exist a meromorphic section of L which is holomorphic on $X\backslash\{p_\infty\}$ and has a pole of order n at p_∞. By Riemann-Roch it can be easily proved that there are exactly g gaps in $\{0, 1, ..., 2g-1\}$ for any (L, p_∞). Let δ be Riemann's constant, $L_{e+\delta}$ the flat line bundle corresponding to the point $e+\delta$ on the Jacobian and

$$b_1 < \cdots < b_g,$$
$$w_1 < \cdots < w_g,$$

the gaps of $L_{e+\delta}$ and L_0 at p_∞ respectively, where L_0 is the trivial line bundle. We define the partition $\lambda = (\lambda_1, ..., \lambda_g)$ by

$$\lambda = (b_g, b_{g-1}, ..., b_1) - (g-1, g-2, ..., 0),$$

and consider the Schur function $s_\lambda(t)$, $t = (t_1, t_2, ...)$. The special property of $s_\lambda(t)$ is that it depends only on t_{w_i}, $1 \leq i \leq g$. This property is crucial when we connect it to the theta function.

To give a relation of $s_\lambda(t)$ with the theta function we need to make a change of variables which is given by a certain non-normalized period matrix. To define it we specify a local coordinate z around p_∞. Then there is a basis du_{w_i}, $1 \leq i \leq g$, of holomorphic one forms which has the expansion at p_∞ of the form

$$du_{w_i} = (z^{w_i - 1} + O(z^{w_i}))dz, \qquad 1 \leq i \leq g.$$

It is not unique. We take any one of them. Let $2\omega_1$ be the $\{\alpha_i\}$ period matrix of $\{du_{w_i}\}$. Let $u = {}^t(u_{w_1}, ..., u_{w_g})$. We assign the weight i to variables u_i and t_i. Then the Schur function $s_\lambda(t)$ becomes a weight-homogeneous polynomial with the weight $|\lambda| = \lambda_1 + \cdots + \lambda_l$ for $\lambda = (\lambda_1, ..., \lambda_l)$.

We prove that the theta function has the expansion of the form

$$C\theta\big((2\omega_1)^{-1}u + e|\Omega\big) = s_\lambda(t)|_{t_{w_i} = u_{w_i}} + \text{higher weight terms}, \tag{1}$$

for some constant C which is given explicitly by a theta constant (see Theorem 10).

Next we study the expansion of the function $\theta(p_1 + \cdots + p_g + e|\Omega)$ in $z_g = z(p_g)$ where p_i inside the theta function denotes the image of p_i by the Abel-Jacobi map with the base point p_∞. To describe the results we need some sequence of numbers which we call a-sequence.

Let $0 \leq b_1^* < b_2^* < \cdots$ be non-gaps of $L_{e+\delta}$ at p_∞. For $0 \leq k \leq g-1$ define m_k by

$$m_k = \sharp\{i \,|\, b_i^* < g-k\}$$

and $a_i^{(k)}$, $1 \leq i \leq m_k$, by

$$(a_1^{(k)}, ..., a_{m_k}^{(k)}) = (b_{g-k}, b_{g-k-1}, ..., b_{g-k-m_k+1}) - (b_1^*, ..., b_{m_k}^*).$$

Any $a_i^{(k)}$ is proved in $\{w_j\}$. In general for a non-empty subset $I=\{i_1,...,i_l\}$ we set

$$\partial_I=\partial_{u_{i_1}}\cdots\partial_{u_{i_l}}, \qquad \partial_{u_i}=\frac{\partial}{\partial u_i}.$$

and set $\partial_I=1$ for $I=\phi$. Let $A_k=\{a_i^{(k)}\}$ for $k\geq 1$ and $A_0=\phi$. We show that the following expansion is valid for $1\leq k\leq g$:

$$\partial_{A_k}\theta(\sum_{i=1}^{k}p_i+e|\Omega)=\tilde{c}_k\partial_{A_{k-1}}\theta(\sum_{i=1}^{k-1}p_i+e|\Omega)z_k^{\lambda_k}+O(z_k^{\lambda_k+1}),$$

where $z_k=z(p_k)$ and $c_k=\pm 1$ is explicitly given (Theorem 12). The non-vanishing of the left hand side follows from (1). This type of expansion was first pointed out in [29] in the case of hyperelliptic curves and $e=-\delta$ and was applied to addition formulae of the fundamental sigma function . The results are extended to the case of (n,s) curves and $e=-\delta$ in [27, 22] and to that of telescopic curves and $e=-\delta$ in [2]. Here we extend the results to the case of an arbitrary Riemann surface and an arbitrary point e on the theta divisor.

The results on the expansions of the theta function above and the way to prove it implies an interesting extension and a refinement of Riemann's singularity theorem.

Riemann's singularity theorem asserts that the multiplicity of $\theta(Z|\Omega)$ at e is m_0 [12, 11]. In other words it says that any derivative of $\theta(Z|\Omega)$ of degree less than m_0 vanishes at e and some derivative of degree m_0 does not vanish at e, where the degree signifies the degree as a differential operator. Here we should notice that the theorem tells nothing on which derivatives do not vanish in general.

We derive the following properties of the theta function from those of τ functions and Schur functions:

(i) $\partial_I\theta(e|\Omega)=0$ for any $I=(i_1,...,i_m)$ if $i_1+\cdots+i_m<|\lambda|$.

(ii) $\partial_I\theta(e|\Omega)=0$ for any $I=(i_1,...,i_m)$ if $m<m_0$.

(iii) $\partial_{A_0}\theta(e|\Omega)\neq 0$.

The properties (ii) and (iii), in particular, implies Riemann's singularity theorem. Moreover we see that the A_0-derivative gives the non-vanishing derivative of degree m_0 explicitly. The vanishing property (i) is a new vanishing property which does not follow from Riemann's singularity theorem. Therefore (i)-(iii) give an extension and a refinement of Riemann's singularity theorem.

Finally this A_0-derivative can be used to define an appropriate normalization in defining the sigma function such that the resulting function becomes modular invariant. In order to define sigma functions we need a certain bilinear meromorphic differential. The normalized bilinear differential $\omega(p_1,p_2)=d_{p_1}d_{p_2}\log E(p_1,p_2)$, where $E(p_1,p_2)$ is the prime form, plays a fundamental role in the theory of theta functions [12]. However $\omega(p_1,p_2)$ depends on the choice of canonical homology basis. Klein modified ω so that it does not depend on the choice of canonical homology basis [12, 14],

which we call Kein form. Let $\widehat{\omega}(p_1, p_2)$ be a bilinear differential which is obtained from the Klein form by adding $\sum c_{ij} du_{w_i} du_{w_j}$, where $\{c_{ij}\}$ are independent of the choice of canonical homology basis and satisfy $c_{ij} = c_{ji}$. The sigma function associated with $(X, \{\alpha_i, \beta_i\}, p_\infty, z, e, \{du_{w_i}\}, \widehat{\omega})$ is defined as follows.

Let us write $e = \Omega\varepsilon' + \varepsilon''$ with $\varepsilon', \varepsilon'' \in \mathbb{R}^g$ and set $\varepsilon = {}^t(\varepsilon', \varepsilon'')$. Using Riemann's theta function with the characteristics ε we set

$$C_e = \partial_{A_0} \theta[\varepsilon](0|\Omega),$$

which does not vanish due to (iii) above. Then we define the sigma function with the characteristics ε by

$$\sigma[\varepsilon](u) = C_e^{-1} \exp(\frac{1}{2} {}^t u \eta_1 \omega_1^{-1} u) \theta[\varepsilon]((2\omega_1)^{-1} u \,|\, \Omega),$$

where η_1 is the period of certain second kind differentials which is computed from the α_i-integral of $\widehat{\omega}$. The part without C_e^{-1} of the right hand side, which we call the main part, is already proposed in [4] without explicit construction of η_1. In [18] Korotkin and Shramchenko proposed to use Klein form to define η_1. They have shown that the main part, multiplied by a certain theta constant which is apparently different from C_e, is invariant under the change of the canonical homology basis up to multiplication of $8N$-th root of unity, where N is the number of non-singular even half periods. We show that the sigma function normalized by C_e is invariant under the action of $Sp(2g, \mathbb{Z})$ on canonical homology basis. We call this property the modular invariance of the sigma function.

There remain several fundamental problems to be solved. We have determined the initial term of the expansion of the theta function with respect to weight. In applications sometimes the initial term with respect to degree is necessary[26]. In [4] the minimal degree term is determined for a hyperelliptic curve with e being certain half periods as certain determinants. In this paper we have determined the minimal degree term in the minimal weight term for arbitrary (X, e). It is interesting to determine the full minimal degree term. To this end it is necessary to study higher weight terms in the expansion of τ-function. The results in this direction can be applied to the study on inversions of hyperelliptic integrals [10].

The relation of Klein form with the bilinear meromorphic differentials of (n, s) curves, telescopic curves and others [6, 4, 1, 24, 16, 17], which are constructed algebraically, should be clarified (see [8] for some examples). It is also interesting to determine the explicit relation between the normalization constants given in this paper and those in [18]. In the case of genus one the relation is given by the celebrated Jacobi's derivative formula.

In the case of an (n, s) curve the coefficients of the series expansion of the fundamental sigma function, which corresponds to $e = -\delta$, are polynomials of the coefficients of the defining equation of the curve [24, 25]. It is known that to any algebraic curve there exists a certain normal form of defining equations [21]. It is expected that the coefficients of the series expansion of the fundamental sigma function of an algebraic

curve can be expressed by the coefficients of defining equations of the curve as in the case of an (n, s) curve. To this end we need to construct the (modified) Klein form algebraically using the defining equation of the curve. This construction is an independent interesting problem.

The paper is organized as follows. In section 2 after the the review on divisors and line bundles on a Riemann surface we define the partition corresponding to a geometric data using gaps of line bundles. The properties of the Schur functions corresponding to geometric data are studied in section 3. In section 4 the properties of the function which has a similar expansion to τ-function of the KP-hierarchy is studied. Sato's theory of the KP-hierarchy is reviewed in section 5. In section 6 the point of UGM corresponding to an algebro-geometric solution of the KP-hierarchy is determined. It is shown that the solution corresponding to $(X, \{\alpha_i, \beta_i\}, p_\infty, z, e)$ is in the cell UGM^λ of UGM with the partition λ corresponding to this geometric data. This is the extension of the result in [15] to the non-generic case. The series expansions of the theta function are studied in section 7. We give examples of partitions corresponding to geometric data here. In the case of the hyperelliptic curve defined by an odd degree polynomial, the partition λ corresponding to any data of the form $(X, \{\alpha_i, \beta_i\}, \infty, e)$ are determined explicitly. In section 8 sigma functions with arbitrary real characteristics are defined and they are shown to be modular invariant.

2 Geometric Data

2.1 Preliminaries

Here we collect necessary facts on Riemann surfaces following mainly [12].

Let X be a compact Riemann surface of genus g and $\{\alpha_i, \beta_i\}$ a canonical homology basis of X. Then the normalized basis $\{dv_i\}$ of holomorphic one forms and the normalized period matrix is determined:

$$\int_{\alpha_j} dv_i = \delta_{ij}, \qquad \Omega = \left(\int_{\beta_j} dv_i \right).$$

Riemann's theta function with characteristics $\varepsilon = {}^t(\varepsilon', \varepsilon'')$, $\varepsilon', \varepsilon'' \in \mathbb{R}^g$ is defined by

$$\theta[\varepsilon](z|\Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i\, {}^t(n + \varepsilon')\Omega(n + \varepsilon') + 2\pi i\, {}^t(n + \varepsilon')(z + \varepsilon'')), \qquad z = {}^t(z_1, ..., z_g).$$

For $\varepsilon = {}^t(0, 0)$, $\theta[\varepsilon](z|\Omega)$ is denoted by $\theta(z|\Omega)$.

Let $J(X) = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ be the Jacobian variety X. By fixing a base point p_∞ we have the Abel-Jacobi map :

$$I : X \to J(X), \qquad I(p) = \int_{p_\infty}^{p} dv,$$

where dv is the vector of normalize holomorphic one forms, $dv = {}^t(dv_1, ..., dv_g)$. The Jacobian $J(X)$ is isomorphic to the group of dvisor classes of degree zero by the Abel-Jacobi map. We sometimes identify a divisor of degree zero with its Abel-Jacobi image in $J(X)$:

$$\sum(p_i - q_i) = \sum (I(p_i) - I(q_i)) .$$

A choice $\{\alpha_i, \beta_i\}$ specifies the Riemann divisor Δ. It is a divisor class of degree $g-1$ and satisfies $2\Delta = K_X$, where K_X is the canonical divisor class of X. Further if a point p_∞ on X is specified, Riemann's constant δ is determined as an element of $J(X)$. It is related to Δ by

$$\Delta - (g-1)p_\infty = \delta.$$

To each divisor D is associated a holomorphic line bundle L_D on X (see [12] for example). If D is of degree zero, then

$$\frac{\theta(p - p_\infty - D - f)}{\theta(p - p_\infty - f)}$$

is a meromorphic section of L_D, where f is a generic point of $\mathbb{C}^g$ such that both denominator and numerator do not vanish identically. In terms of the transformation law, if D is represented by $c = {}^t(c_1, ..., c_g) \in \mathbb{C}^g$ in $J(X)$, a section F of L_D satisfies

$$F(p + \alpha_j) = F(p), \quad F(p + \beta_j) = e^{2\pi i c_j} F(p). \tag{2}$$

In this case L_D is also denoted by L_c. A different choice of the representative c of D gives a holomorphically equivalent line bundle. The holomorphic line bundle corresponding to a divisor of degree zero is called a flat line bundle since it admits a holomorphic flat connection.

To each holomorphic line bundle there corresponds the sheaf of germs of holomorphic sections of it. We shall introduce some notation related with this sheaf.

For two positive divisors A, B and a holomorphic line bundle L we denote by $L(B - A)$ the sheaf of germs of meromorphic sections of L whose poles are at most at B and whose zrors are at least at A. Let $\mathcal{O}$ denote the sheaf corresponding to the trivial line bundle L_0. Then we have

$$L_D \simeq \mathcal{O}(D). \qquad L_D(B - A) \simeq \mathcal{O}(D + B - A), \tag{3}$$

as sheaves of $\mathcal{O}$ modules.

We set

$$h^0(L(B - A)) = \dim H^0(X, L(B - A)).$$

By (3) we have

$$h^0(L_D(B - A)) = h^0\big(\mathcal{O}(D + B - A)\big).$$

We denote the right hand side of this equation by $h^0(D + B - A)$.

For a point p of X let $L(*p)$ denote the sheaf of germs of meromorphic sections of L which have a pole of any order at p.

2.2 Gaps

Definition 1 *A non-negative integer b is called a gap of a flat line bundle L at p_∞ if there does not exist a meromorphic section of L which is holomorphic outside p_∞ and has a pole of order b at p_∞. A non-negative integer which is not a gap is called a non-gap of L at p_∞.*

Lemma 1 *There are exactly g gaps for any pair (L, p_∞) of a flat line bundle L and a point p_∞ of X.*

Proof. Let L correspond to the divisor D. By the Riemann-Roch theorem we have

$$h^0(D + np_\infty) - h^0(K_X - D - np_\infty) = 1 - g + n. \tag{4}$$

If $n \geq 2g - 1$,

$$h^0(K_X - D - np_\infty) = 0,$$

since the degree of $K_X - D - n$ is negative. In particular

$$h^0(D + (2g - 1)p_\infty) = g.$$

Since we have $2g$ integers between 0 and $2g - 1$, it means that there are g gaps in $\{0, 1, ..., 2g - 1\}$. It is obvious that there are no other gaps by (4). □

2.3 Partition associated with a geometric data

Let e be a zero of the theta function $\theta(z \,|\, \Omega)$. By Riemann's vanishing theorem e can be written as

$$e = q_1 + \cdots + q_{g-1} - \Delta, \tag{5}$$

for some $q_1, ..., q_{g-1} \in X$.

We consider the divisor $e + \delta$ of degree zero:

$$e + \delta = q_1 + \cdots + q_{g-1} - (g - 1)p_\infty. \tag{6}$$

The proof of Lemma 1 shows that there are exactly g gaps and g non-gaps of (L, p_∞) in $\{0, 1, ..., 2g - 1\}$

We introduce two kinds of gaps and non-gaps simultaneously. Let

$$b_1 < b_2 < \cdots < b_g,$$
$$b_1^* < b_2^* < \cdots$$

be gaps and non-gaps of $L_{e+\delta}$ at p_∞ and

$$w_1 < w_2 < \cdots < w_g,$$
$$w_1^* < w_2^* < \cdots$$

those of L_0 at p_∞.

Remark The sequence $(w_1, ..., w_g)$ is the gap sequence of a Riemann surface X at p_∞ [11]. A point p_∞ for which $(w_1, ..., w_g) \neq (1, ..., g)$ is called a Weierstrass point. We have $w_1 = 1$ and $w_1^* = 0$.

Definition 2 *We define the partion $\lambda = (\lambda_1, ..., \lambda_g)$ associated with $(X, \{\alpha_i, \beta_i\}, p_\infty, e)$ by*

$$\lambda = (b_g, ..., b_1) - (g-1, ..., 1, 0). \tag{7}$$

In the next section we study the properties of the Schur function corresponding to λ.

3 Schur function of a Riemann surface

3.1 Dependence on variables

The Schur function $s_\mu(t)$, $t = (t_1, t_2, t_3, ..)$, [20] corresponding to a partition $\mu = (\mu_1, ..., \mu_l)$ is defined by

$$s_\mu(t) = \det\left(p_{\mu_i - i + j}(t)\right)_{1 \leq i,j \leq l},$$

where $p_n(t)$ is the polynomial defined by

$$\exp\Big(\sum_{n=1}^{\infty} t_n k^n\Big) = \sum_{n=0}^{\infty} p_n(t) k^n, \qquad p_n(t) = 0 \text{ for } n < 0.$$

We identify a partition $\mu = (\mu_1, ..., \mu_l)$ with $(\mu_1, ..., \mu_l, 0^i)$ for any i. The Schur function $s_\mu(t)$ does not depend on the choice of i.

We define the weight and degree of the variable t_i to be i and 1 respectively:

$$\mathrm{wt}\,(t_i) = i, \qquad \deg(t_i) = 1.$$

With respcet to weight the Schur function $s_\mu(t)$ is a homogeneous polynomial with the weight $|\mu| = \mu_1 + \cdots + \mu_l$, while it is not homogeneous with respect to degree in general.

By the definition $s_\mu(t)$ is a polynomial of $t_1, ..., t_{\mu_1 + l - 1}$. However the Schur function corresponding to a geometric data depends on fewer number of variables.

Proposition 1 *Let λ be the partition associated with $(X, \{\alpha_i, \beta_i\}, p_\infty, e)$. Then $s_\lambda(t)$ is a polynomial of $t_{w_1}, ..., t_{w_g}$.*

Proof. Notice that the space $H^0(X, L_{e+\delta}(*p_\infty))$ is a $H^0(X, \mathcal{O}(*p_\infty))$-module. Therefore if we define $N_1 = \{w_i^* | i \geq 1\}$ and $N_2 = \{b_i^* | i \geq 1\}$, then N_1 acts on N_2 by addition. Namely for any i, j we have

$$w_i^* + b_j^* \in N_2.$$

It follows that, if $b_j - w_i^* \geq 0$ then it is a gap of L at p_∞. In fact if it is not the case, $b_j - w_i^* \in N_2$. Then $b_j \in N_2 + w_i^* \subset N_2$ which is absurd. The proposition can be proved in a similar manner to Proposition 2 of [27] using this property. □

3.2 a-sequence

Definition 3 *For an integer k such that $0 \le k \le g-1$ we define the integer m_k by*

$$m_k = h^0(q_1 + \cdots + q_{g-1} - kp_\infty).$$

It is possible to describe m_k in terms of gaps or non-gaps of $L_{e+\delta}$.

Lemma 2 *We have*

$$m_k = \sharp\{i \,|\, b_i^* < g-k\} = g-k-\sharp\{i \,|\, b_i < g-k\}.$$

proof. By the definition of $L_{e+\delta}$ we have

$$m_k = h^0\big(L_{e+\delta}((g-k-1)p_\infty)\big),$$

which proves the first equation of the lemma. Since gaps and non-gaps are complements to each other in $\{0,1,...,g-k-1\}$ the second equality follows. $\square$

In order to describe more detailed properties of $s_\lambda(t)$ we need

Definition 4 *For $0 \le k \le g-1$ define the sequence $A_k = (a_1^{(k)}, ..., a_{m_k}^{(k)})$ by*

$$A_k = (b_{g-k}, b_{g-k-1}, ..., b_{g-k-m_k+1}) - (b_1^*, ..., b_{m_k}^*).$$

The sequence A_k is referred to as a-sequence.

For a partition $\mu = (\mu_1, ..., \mu_l)$ and $0 \le k \le l-1$ we set

$$N_{\mu,k} = \sum_{i=k+1}^{l} \mu_i.$$

In the case $k=0$, $N_{\mu,0} = \sum_{i=1}^{l} \mu_i = |\mu|$ is the weight of μ.

Lemma 3 *Suppose that $m_k > 0$. Then*
(i) $a_1^{(k)} > \cdots > a_{m_k}^{(k)}$.
(ii) $a_i^{(k)} \in \{w_j\}$ for any i.
(iii) $\displaystyle\sum_{i=1}^{m_k} a_i^{(k)} = N_{\lambda,k}$.

Proof. (i) is obvious from the definition of $a_i^{(k)}$. Let us prove (ii). We first show that $a_{m_k}^{(k)} > 0$. By Lemma 2 we have $b_1^* < \cdots < b_{m_k}^* < g-k$. Since $\{b_i^*\}$ and $\{b_i\}$ are complement in the set of nonnegative integers to each other,

$$\{b_1^*, ..., b_{m_k}^*\} \sqcup \{b_1, ..., b_{g-k-m_k}\} = \{0,1,...,g-k-1\}. \tag{8}$$

Thus $b^*_{m_k} < g-k \le b_{g-k-m_k+1}$ and $a^{(k)}_{m_k} = b_{g-k-m_k+1} - b^*_{m_k} > 0$. Now, suppose that $a_i^{(k)} = b_{g-k+1-i} - b_i^* \notin \{w_j\}$. Since $a_i^{(k)} > 0$, we have $a_i^{(k)} = w_j^*$ for some j. Then $b_{g-k+1-i} = b_i^* + w_j^*$ is a non-gap of $L_{e+\delta}$, which is impossible. Thus the assertion (ii) is proved.
(iii): We have

$$\sum_{i=1}^{m_k} a_i^{(k)} = \sum_{i=g-k-m_k+1}^{g-k} b_i - \sum_{i=1}^{m_k} b_i^* = \sum_{i=1}^{g-k} b_i - \sum_{i=1}^{g-k-1} i, \tag{9}$$

where we use (8). Since $\lambda_i = b_{g+1-i} - (g-i)$, the right hand side of (9) equals to $\lambda_{k+1} + \cdots + \lambda_g$. $\Box$

3.3 Vanishing and non-vanishing

We introduce the analogue of the Abel-Jacobi map for Schur function.

Definition 5 *Define* $[x]$ *by*

$$[x] = (x, \frac{x^2}{2}, \frac{x^3}{3}, ...).$$

Using the a-sequence we can describe the properties of derivatives of $s_\lambda(t)$.

For a sequence $I = (i_1, ..., i_r)$ of positive integers we set

$$\partial_{t,I} = \partial_{t_{i_1}} \cdots \partial_{t_{i_r}}, \qquad \partial_{t_i} = \frac{\partial}{\partial t_i}.$$

In the following we sometimes use the expressions $\sum_{i=1}^k [x_i]$ and $s_{(\mu_1,...,\mu_k)}(t)$ for $k=0$. They should be understood as

$$\sum_{i=1}^k [x_i] = 0, \qquad s_{(\mu_1,...,\mu_k)}(t) = 1.$$

Theorem 1 *Suppose* $m_k > 0$. *Let* λ *be the partition defined by (7) and* $\mu = (\mu_1, ..., \mu_l)$ *a partition satisfying* $\mu_i = \lambda_i$ *for* $i \ge k+1$. *Then*

$$\partial_{t,A_k} s_\mu (\sum_{i=1}^k [x_i]) = c_k s_{(\mu_1,...,\mu_k)}(\sum_{i=1}^k [x_i]),$$

where

$$c_k = sgn \begin{pmatrix} b_1^* & \cdots & b^*_{m_k} & b_{g-k-m_k} & \cdots & b_1 \\ g-k-1 & \cdots & \cdot & \cdot & \cdots & 1 \end{pmatrix}. \tag{10}$$

Proof. The theorem can be proved in a completely similar way to Theorem 1 in [27]. □

For two partitions $\mu = (\mu_1, ..., \mu_l)$, $\nu = (\nu_1, ..., \nu_{l'})$ we define $\mu \le \nu$ by the condition $\mu_i \le \nu_i$ for any i.

Then we have the following vanishing theorem for Schur functions.

Theorem 2 *Let λ be the partition defined by (7). Then*

(i) Let μ be any partition. Then, for any sequence $I = (i_1, ..., i_m)$, $m \ge 1$ satisfying $\sum_{j=1}^{m} i_j \ne |\mu|$ we have

$$\partial_{t,I} s_\mu(0) = 0, \tag{11}$$

(ii) Let μ be a partition satisfying $\mu \ge \lambda$. If $m < m_0$ we have

$$\partial_{t,I} s_\mu(0) = 0, \tag{12}$$

for any and $I = (i_1, ..., i_m)$.

(iii) $\partial_{t,A_0} s_\lambda(0) = c_0$, where $c_0 = \pm 1$ is given by (10).

Proof. Since $s_\mu(t)$ is weight-homogeneous with the weight $|\mu|$, (i) is obvious. (iii) is the case of $k = 0$ of Theorem 1. So let us prove (ii).

If $\sum_{j=1}^{m} i_j \ne |\mu|$, the left hand side of (12) vanishes by (i). So we assume $\sum_{j=1}^{m} i_j = |\mu| = N_{\mu,0}$.

Let $[i_1, ..., i_l]$ be the determinant of the $l \times l$ matrix whose j-th row is given by

$$(p_{i_j - l+1}(t), ..., p_{i_j-1}(t), p_{i_j}(t)).$$

We write $[i_1, ..., i_l](t)$ if it is necessary to indicate t. Let us define the strictly decreasing sequence $(b'_l, ..., b'_1)$ corresponding to $\mu = (\mu_1, ..., \mu_l)$ by

$$(b'_l, ..., b'_1) = (\mu_1, ..., \mu_l) + (l-1, ..., 1, 0).$$

We take $l \ge g$ by inserting several 0's to the end of μ if necessary. Then the condition $\mu \ge \lambda$ is

$$b'_i \ge b_{i-l+g} + l - g, \qquad l - g + 1 \le i \le l.$$

With this notation we have

$$s_\mu(t) = [b'_l, ..., b'_1].$$

Since $\partial_{t,i} p_j(t) = p_{j-i}(t)$ and the derivative of the determinant by $\partial_{t,i}$ is the sum of the determinant whose j-th row is differentiated, we have

$$\partial_{t,i} s_\mu(t) = \sum_{j=1}^{l} [b'_l, ..., b'_j - i, ..., b'_1].$$

Thus the left hand side of (12) is written as a sum of the determinants of the form

$$[b'_l - r_l, ..., b'_1 - r_1](0). \tag{13}$$

If $r_j > 0$ then it means that the j-th row is differentiated at least once.

We show that all terms (13) appearing in the left hand side of (12) vanish.

Suppose that there is a non-zero term (13). Since the number of the derivatives in the left side of (12) is $m < m_0$, some row among m_0 rows labeled by $l - m_0 + 1, ..., l$ is not differentiated. We call this row the j-th row. By (8) we have

$$b_1 < \cdots < b_{g-m_0} < g \le b_{g-m_0+1} < \cdots < b_g.$$

Thus

$$b'_j \ge b_{j-l+g} + l - g \ge l,$$

since $g - m_0 + 1 \le j - l + g \le g$ for $l - m_0 + 1 \le j \le l$. By Lemma 2 (ii) in [27] such a term is zero, which contradicts the assumption. □

Remark The assertions (ii) and (iii) of Theorem 2 is an analogue of Riemann's singularity theorem for Schur functions.

3.4 Minimal degree term

Assertions (ii) and (iii) of Theorem 2 mean that the term $t_{a_1^{(0)}} \cdots t_{a_{m_k}^{(0)}}$ is one of monomials with the minimal degree which appear in $s_\lambda(t)$. In fact it is possible to determine the minimal degree term of $s_\lambda(t)$.

Let $\mu = (\mu_1, ..., \mu_l)$ be a partition and $L_\mu(t)$ the minimal degree term of $s_\mu(t)$:

$$s_\mu(t) = L_\mu(t) + \text{higher degree terms}.$$

Define $m_0(\mu)$ by

$$m_0(\mu) = l - \sharp\{\, i \,|\, b'_i < l \,\}, \qquad (b'_l, ..., b'_1) = (\mu_1, ..., \mu_l) + (l-1, ..., 1, 0).$$

By Lemma 2 we have $m_0(\lambda) = m_0$.

Proposition 2 *We have*

$$L_\mu(t) = (-1)^{N_{\mu,m_0(\mu)}} \det(t_{\mu_i - i + j})_{1 \le i \le m_0(\mu), j \ne l-b'_1, ..., l-b'_{l-m_0(\mu)}}.$$

In particular the degree of $L_\mu(t)$ is $m_0(\mu)$.

Proof. Since

$$p_n(t) = t_n + \text{higher degree terms}$$

we have

$$s_\mu(t) = \det(t_{\mu_i - i + j})_{1 \le i,j \le l} + \text{higher degree terms}, \tag{14}$$

where we set $t_0 = 1$ and $t_i = 0$ for $i < 0$. Consider the determinant in the right hand side of (14). Consider i with $b'_i < l$. The i-th row from the bottom of the matrix $(t_{\mu_i - i + j})_{1 \le i,j \le l}$ is

$$(0, ..., 0, t_0, t_1, ..., t_{b'_i}).$$

We first expand the determinant in the l-th row, which corresponds to $i = 1$, and pick up the term containing t_0. It is the determinant of the matrix which is obtained by removing l-th row and $(l - b'_1)$-th column times $(-1)^{l+(l-b'_1)}$. We proceed similarly for $(l-1)$-th row, $(l-2)$-th row, ..., $(m_0(\mu)+1)$-th row. Then we get

$$s_\mu(t) = (-1)^{N_{\mu, m_0(\mu)}} \det(t_{\mu_i - i + j})_{1 \le i \le m_0(\mu),\ j \ne l - b'_1, ..., l - b'_{l - m_0(\mu)}} + \text{higher degree terms.}$$

What we have to check is that the first term of the right hand side is not identically zero. Notice the monomial in t_i's in the determinant which is obtained by taking the product of anti-diagonal components. This monomial is unique among the $m_0(\mu)!$ terms in the expansion of the determinant and has ± 1 as its coefficient. $\square$

Example Consider a hyperelliptic curve of genus g defined by $y^2 = f(x)$, where $f(x)$ is a polynomial of degree $2g+1$ without multiple zeros and take $p_\infty = \infty$, $e = -\delta$. Then $\lambda = (g, g-1, ..., 1)$. In this case $m_0 = [\frac{g+1}{2}]$, $N_{\lambda, m_0} = (1/2)(g - m_0)(g + 1 - m_0)$ and

$$L_\lambda(t) = (-1)^{N_{\lambda, m_0}} \det(t_{2k+1-2i+2j})_{1 \le i,j \le k} \text{ for } g = 2k,$$

$$L_\lambda(t) = (-1)^{N_{\lambda, m_0}} \det(t_{2k+1-2i+2j})_{1 \le i,j \le k+1} \text{ for } g = 2k+1.$$

If we introduce the variables with different indices by

$$(u_1, u_2, ..., u_g) = (t_{2g-1}, t_{2g-3}, ..., t_1),$$

then

$$L_\lambda(t) = (-1)^{(1/2)g(g+1) + g m_0} \begin{vmatrix} u_1 & u_2 & \cdots & u_{m_0} \\ u_2 & u_3 & \cdots & u_{m_0+1} \\ \vdots & & & \vdots \\ u_{m_0} & u_{m_0+1} & \cdots & u_{2m_0-1} \end{vmatrix},$$

which is precisely the Hankel determinant formula for the minimal degree term of the series expansion of the hyperelliptic theta function derived in [4, 3].

4 Tau function

4.1 Expansion on Abel-Jacobi images

Let λ be the partition (7) associated with $(X, \{\alpha_i, \beta_i\}, p_\infty, e)$. In this section we consider an arbitrary function of the form

$$\tau(t) = s_\lambda(t) + \sum_{\lambda < \mu} \xi_\mu s_\mu(t).$$

For $1 \leq k \leq g$ we set

$$\tau^{(k)}(t) = s_{(\lambda_1,...,\lambda_k)}(t) + \sum_{\mu} \xi_\mu s_{(\mu_1,...,\mu_k)}(t),$$

where the sum in the right hand side runs over partitions $\mu = (\mu_1, ..., \mu_g)$ satisfying the conditions $\lambda < \mu$, $\mu_i = \lambda_i$ for $k+1 \leq i \leq g$. We set $\tau^{(0)}(t) = 1$.

Theorem 3 *Suppose that $m_k > 0$. Let c_k be given by (10). Then*

(i) $\partial_{t,A_k} \tau \left(\sum_{i=1}^{k} [x_i] \right) = c_k \tau^{(k)} \left(\sum_{i=1}^{k} [x_i] \right).$

(ii) $\tau^{(k)} \left(\sum_{i=1}^{k} [x_i] \right) = \tau^{(k-1)} \left(\sum_{i=1}^{k-1} [x_i] \right) x_k^{\lambda_k} + O(x_k^{\lambda_k+1}).$

(iii) $\partial_{t,A_k} \tau \left(\sum_{i=1}^{k} [x_i] \right) = \frac{c_k}{c_{k-1}} \partial_{t,A_{k-1}} \tau \left(\sum_{i=1}^{k-1} [x_i] \right) x_k^{\lambda_k} + O(x_k^{\lambda_k+1}).$

Proof. The proof of the theorem is similar to that of Theorem 5 in [27]. For the sake of completeness we give a proof here.

Let $\mu = (\mu_1, ..., \mu_l)$ be a partition satisfying $\lambda \leq \mu$. Here l is not necessarily the length of μ. By (iii) of Lemma 3 we have

$$\sum_{i=1}^{m_k} a_i^{(k)} = \sum_{i=k+1}^{g} \lambda_i \leq \sum_{i=k+1}^{l} \mu_i = N_{\mu,k}.$$

In case the last inequality is a strict inequality

$$\partial_{t,A_k} s_\mu \left(\sum_{i=1}^{k} [x_i] \right) = 0 \tag{15}$$

by Proposition 3 in [27]. Thus, if the left hand side of (15) is not zero we have

$$\sum_{i=k+1}^{g} \lambda_i = \sum_{i=k+1}^{l} \mu_i$$

which implies

$$\lambda_i = \mu_i, \quad k+1 \leq i \leq g,$$

and $\mu_i = 0$ for $i > g$, since $\lambda \leq \mu$. Then we have, by Theorem 1,

$$\partial_{t,A_k} s_\mu \left(\sum_{i=1}^{k} [x_i] \right) = c_k s_{(\mu_1,\ldots,\mu_k)} \left(\sum_{i=1}^{k} [x_i] \right).$$

The assertion (i) follows from this. The assertion (ii) is already proved in (ii) of Theorem 5 in [27], since no specialty of the partition λ associated with an (n, s) curve is used there. (iii) follows from (i) and (ii). $\square$

4.2 Vanishing and non-vanishing

The following vanishing theorem for the function τ is valid.

Theorem 4 *(i) For any $I = (i_1, ..., i_m)$, $m \geq 1$, satisfying $\sum_{j=1}^{m} i_j < |\lambda|$ we have*

$$\partial_{t,I} \tau(0) = 0.$$

(ii) If $m < m_0$

$$\partial_{t,I} \tau(0) = 0,$$

for any $I = (i_1, ..., i_m)$.

(iii) $\partial_{t,A_0} \tau(0) = c_0$, where $c_0 = \pm 1$ is given by (10).

Proof. The assertions of the theorem follow from (ii), (iii) of Theorem 2 and the definition of the function $\tau(t)$. $\square$

Notice that this theorem is an analogue of Riemann's singularity theorem for the function $\tau(t)$.

5 Sato's theory on soliton equations

In this section we review Sato's theory of the KP-hierarchy[34] which makes a one to one correspondence between solutions of the KP-hierarchy and points of an infinite dimensional Grassmann manifold, called the universal Grassmann manifold (UGM).

5.1 The KP-hierarchy

The KP-hierarchy is the infinite system of differential equations for $\tau(t)$ given by

$$\int \tau(t - s - [k^{-1}]) \tau(t + s + [k^{-1}]) e^{-2\sum_{i=1}^{\infty} s_i k^i} dk = 0 \tag{16}$$

where $t = {}^t(t_1, t_2, , , ,)$, $s = {}^t(s_1, s_2, ...)$ and the integral signifies taking residue at $k = \infty$ [7]. Namely we formally expand the integrand in the series of k and y and equate the coefficient of $k^{-1}s_1^{\gamma_1}s_2^{\gamma_2}\cdots$ to zero. Then we get an infinite number of differential equations for $\tau(t)$. In particular taking the coefficient of $k^{-1}s_3$ we get the bilinear form of the Kadomtsev-Petviashvili (KP) equation:

$$(D_1^4 + 3D_2^2 - 4D_1D_3)\tau(t)\cdot\tau(t) = 0,$$

where D_i is the Hirota derivative defined by

$$\tau(t+s)\tau(t-s) = \sum(D_1^{\gamma_1}D_2^{\gamma_2}\cdots)\tau(t)\cdot\tau(t)\ \frac{s_1^{\gamma_1}s_2^{\gamma_2}\cdots}{\gamma_1!\gamma_2!\cdots}. \tag{17}$$

The initial value problem of the KP-hierarchy is uniquely solvable and the set of the initial values forms the infinite dimensional Grassmann manifold UGM.

5.2 UGM

Let us give the definition of UGM [34]. To this end we consider the ring of microdifferential operators with the coefficients in the formal power series ring $R = \mathbb{C}[[x]]$ in one variable x. Namely $\mathcal{E}_R$ consists of all expressions of the form

$$a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \cdots, \qquad n\in\mathbb{Z}, \quad a_i(x)\in R,$$

where $\partial = \partial/\partial x$. Using

$$a\partial^n = \sum_{i=0}^{\infty}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)\partial^{n-i}a^{(i)}, \qquad a^{(i)} = \frac{d^ia}{dx^i},$$

elements of $\mathcal{E}_R$ can equally be rewritten in the form

$$\sum_{i\le n}\partial^i b_i, \qquad n\in\mathbb{Z}, \qquad b_i\in R. \tag{18}$$

Next we intoduce the left $\mathcal{E}_R$-module V by

$$V = \mathcal{E}_R/\mathcal{E}_R x.$$

The expression (18) implies that V is isomorphic to the space of microdifferential operators with constant coefficients:

$$V \simeq \mathbb{C}((\partial^{-1})). \tag{19}$$

Then we see that V has the decomposition of the form

$$V = V_\phi \oplus V_0, \qquad V_\phi = \mathbb{C}[\partial], \qquad V_0 = \mathbb{C}[[\partial^{-1}]]\partial^{-1}.$$

Let us define the element e_i of V by

$$e_i = \partial^{-i-1} \quad \mathrm{mod.}\, \mathcal{E}_R x, \qquad i \in \mathbb{Z}.$$

Then

$$V_\phi = \oplus_{i=-\infty}^{-1} \mathbb{C} e_i, \qquad V_0 = \prod_{i=0}^{\infty} \mathbb{C} e_i.$$

The action of $\mathcal{E}_R$ on V is given by

$$\partial^{\pm 1} e_i = e_{i \mp 1}, \qquad x e_i = (i+1) e_{i+1}.$$

UGM is defined as the set of subspaces of V which are comparable with V_ϕ. To be precise we need some notation. For a subspace U of V we denote π_U the map

$$\pi_U : U \longrightarrow V/V_0 \simeq V_\phi,$$

which is obtained as the composition of the inclusion $U \hookrightarrow V$ and the natural projection $V \to V/V_0$.

Definition 6 *The universal Grassmann manifold UGM is the set of subspaces $U \subset V$ such that $Ker \pi_U$ and $Coker \pi_U$ are of finite dimension and satisfy*

$$\dim(Ker \pi_U) - \dim(Coker \pi_U) = 0.$$

A point U of UGM is specified by giving a frame of U, which is an ordered basis of U.

To each point U of UGM there exists the unique sequence of integers $\rho = (\rho(i))_{i<0}$ and the unique frame $\boldsymbol{\xi} = (\xi_j)_{j<0}$, $\xi_j = \sum_{i \in \mathbb{Z}} \xi_{ij} e_i$, of U such that

$$\begin{aligned} &\rho(-1) > \rho(-2) > \cdots, \qquad \rho(i) = i \text{ for } i << 0, \\ &\xi_{ij} = \begin{cases} 0 & i < \rho(j) \text{ or } i = \rho(j') \text{ for some } j' > j \\ 1 & i = \rho(j). \end{cases} \end{aligned} \tag{20}$$

This frame $\boldsymbol{\xi}$ is called the normalized frame of U.

The dimensions of $\mathrm{Ker}\, \pi_U$ and $\mathrm{Coker}\, \pi_U$ are expressed by ρ:

$$\dim(\mathrm{Ker}\, \pi_U) = \sharp\{ i \,|\, \rho(i) \geq 0 \}, \qquad \dim(\mathrm{Coker}\, \pi_U) = \sharp\{ i < 0 \,|\, i \notin \{\rho(j)\} \}.$$

In general for a sequence $\rho = (\rho(i))_{i<0}$ satisfying the condition (20) define the partition λ_ρ by

$$\lambda_\rho = (\lambda_{\rho,1}, \lambda_{\rho,2}, ...) = (\rho(-1), \rho(-2), ...) + (1, 2, ...). \tag{21}$$

The partition $\lambda = (0, 0, ...)$ corresponding to $\rho = (-1, -2, ...)$ is denoted by ϕ.

Conversely for any partition $\lambda = (\lambda_1, ..., \lambda_l)$ one can construct ρ satisfying (20) using (21), where we set $\lambda_i = 0$ for $i > l$.

For a partition λ let UGM^λ be the set of points U of UGM such that the partition associated with U is λ. Then UGM has the decomposition

$$UGM = \sqcup_\lambda UGM^\lambda,$$

where λ runs over all partitions.

5.3 Fundamental theorems of Sato's theory

Let U be a point of UGM, $\boldsymbol{\xi} = (\xi_j)_{j<0}$ the normalized frame of U and μ an arbitrary partition. Let us write $\mu = \lambda_\rho$ with ρ satisfying (20). We define the Plücker coordinate ξ_μ of U associated with μ by

$$\xi_\mu = \det\left(\xi_{\rho(i),j}\right)_{i,j<0}.$$

If $U \in UGM^\lambda$ the Plücker coordinates satisfy

$$\xi_\mu = \begin{cases} 0 & \text{unless } \mu \geq \lambda \\ 1 & \mu = \lambda. \end{cases} \tag{22}$$

Definition 7 *Define the tau function corresponding to U by*

$$\tau(t;U) = \sum_\mu \xi_\mu s_\mu(t). \tag{23}$$

Theorem 5 *[34, 33] For any U, $\tau(t;U)$ is a solution of the KP-hierarchy. Conversely any formal power series solution $\tau(t)$ of the KP-hierarchy there exists a unique point U of UGM such that $\tau(t) = c\tau(t;U)$ for some constant c.*

Notice that, for a point U of UGM^λ, we have

$$\tau(t;U) = s_\lambda(t) + \sum_{\lambda<\mu} \xi_\mu s_\mu(t) \tag{24}$$

due to (22).

Now we explain how to recover U from $\tau(t)$.

Let $K = \mathbb{C}((x))$ be the field of formal Laurent series in x and $\mathcal{E}_K = K((\partial^{-1}))$ the ring of microdifferential operators with the coefficients in K.

Definition 8 *Let $\mathcal{W}$ be the set of W in $\mathcal{E}_K$ of the form*

$$W = \sum_{i\leq 0} w_i \partial^i, \quad w_0 = 1,$$

which satisfies the condition

$$x^m W, \quad W^{-1}x^m \in \mathcal{E}_R, \tag{25}$$

for some non-negative integer m.

Then

Theorem 6 *[34, 33] Let W be an element of $\mathcal{W}$ and m an integer as in (25). Then*

$$\gamma(W) = W^{-1}x^m V^\phi$$

defines a bijection $\gamma : \mathcal{W} \to UGM$.

We remark that the map γ does not depend on the choice of m, since $x : V_\phi \to V_\phi$ is a surjection.

Given a formal power series solution of the KP-hierarchy $\tau(t) \neq 0$ the wave function $\bar{\Psi}(t; z)$ and the adjoint wave function $\Psi(t, z)$ are defined by

$$\bar{\Psi}(t; z) = \frac{\tau(t - [z])}{\tau(t)} \exp(\sum_{i=1}^{\infty} t_i z^{-i}).$$

$$\Psi(t; z) = \frac{\tau(t + [z])}{\tau(t)} \exp(-\sum_{i=1}^{\infty} t_i z^{-i}),$$

Let

$$\frac{\tau(t - [z])}{\tau(t)} = \sum_{i=0}^{\infty} w_i z^i, \qquad W = \sum_{i=0}^{\infty} w_i \partial^{-i}.$$

Then

$$\bar{\Psi}(t, z) = W \exp(\sum_{i=1}^{\infty} t_i z^{-i}),$$

where we set $t_1 = x$. The equation (16) implies that Ψ can be written as

$$\Psi(t, z) = (W^*)^{-1} \exp(-\sum_{i=1}^{\infty} t_i z^{-i}).$$

where $P^* = \sum(-\partial)^i a_i(x)$ is the formal adjoint of $P = \sum a_i(x)\partial^i$. We have $(P^*)^{-1} = (P^{-1})^*$ for an invertible $P \in \mathcal{E}_K$ [7].

If $\tau(t)$ is not identically 0, $\tau(x, 0, 0, ...)$ is not identically zero [35] (see Lemma 4 in [25]). Let m_0 be the order of zeros of $\tau(x, 0, 0...)$ at $x = 0$ and $m \geq m_0$. Obviously we have

$$x^m W(x, 0, ...), \quad W(x, 0, ...)^{-1} x^m \in \mathcal{E}_R$$

which implies $W \in \mathcal{W}$. Then

Theorem 7 *[33, 34] There is a constant C such that*

$$C\tau(t) = \tau\left(t; \gamma\left(W(x, 0, ...)\right)\right).$$

The image $\gamma\left(W(x, 0, ...)\right)$ can be computed from $\Psi(t; z)$ using the following proposition [25] (see also [32][15]).

Proposition 3 *Let*

$$x^m \Psi(x, 0, ...; z) = \sum_{i=0}^{\infty} \Psi_i(z) \frac{x^i}{i!}. \tag{26}$$

Then we have, for $i \geq 0$,

$$W(x,0,...)^{-1}x^m e_{-1-i} = (-1)^i \Psi_i(\partial^{-1})e_{-1}.$$

In particular

$$\gamma(W(x,0,...)) = Span_{\mathbb{C}}\{\Psi_i(\partial^{-1})e_{-1} \,|\, i \geq 0\,\},$$

where $Span_{\mathbb{C}}\{\cdots\}$ signifies the vector space spanned by $\{\cdots\}$.

6 Algebro-geometric solution

In this section we determine the point of UGM corresponding to a theta function solution of the KP-hierarchy. We assume that a data $(X, \{\alpha_i, \beta_i\}, p_\infty, z)$ is given, where $(X, \{\alpha_i, \beta_i\}, p_\infty)$ is as before and z is a local coordinate at p_∞.

6.1 Prime form

Let us first recall the prime form [12]. It is known that there exists a non-singular odd half period $e' \in J(X)$. We write e' as

$$e' = q'_1 + \cdots + q'_{g-1} - \Delta, \qquad q'_i \in X.$$

Let $\varepsilon_0 = {}^t(\varepsilon'_0, \varepsilon''_0)$, $\varepsilon'_0, \varepsilon''_0 \in \mathbb{R}^g$ be the characteristics of e':

$$e' = \Omega\varepsilon'_0 + \varepsilon''_0.$$

Then the zero divisor of the holomorphic one form

$$\sum_{i=1}^{g} \frac{\partial\theta[\varepsilon_0]}{\partial z_i}(0)dv_i$$

is $2\sum_{i=1}^{g-1} q'_i$. Since e' is non-singular, there exists a unique, up to constant multiples, holomorphic section of $L_{e'} \otimes L_\Delta$ which vanishes exactly at $q'_1 + \cdots + q'_{g-1}$.

Let h_{ε_0} be a holomorphic section of $L_{e'} \otimes L_\Delta$ such that

$$h_{\varepsilon_0}^2 = \sum_{i=1}^{g} \frac{\partial\theta[\varepsilon_0]}{\partial z_i}(0)dv_i.$$

Then the prime form $E(p_1, p_2)$ is defined by

$$E(p_1,p_2) = \frac{\theta[\varepsilon_0](\int_{p_1}^{p_2} dv)}{h_{\varepsilon_0}(p_1)h_{\varepsilon_0}(p_2)}. \tag{27}$$

Let π_i be the projection of $X \times X$ to the i-th component X and I_{21} the map from $X \times X$ to $J(X)$ defined by $I_{21}(p_1,p_2) = I(p_2) - I(p_1)$. We denote by Θ the holomorphic line bundle on $J(X)$ of which $\theta(z|\Omega)$ is a holomorphic section.

Then the prime form is a holomorphic section of $\pi_1^* L_\Delta^{-1} \otimes \pi_2^* L_\Delta^{-1} \otimes I_{21}^*(\Theta)$. The prime form $E(p_1, p_2)$ is skew symmetric in (p_1, p_2) and vanishes at the diagonal $\{(p,p)|p \in X\}$ to the first order. For other properties see [12].

We need the object which is obtained from the prime form by restricting one of the variables to a point. Using the local coordinate z around p_∞ let us write

$$E(p_1, p_2) = \frac{E(z_1, z_2)}{\sqrt{dz_1}\sqrt{dz_2}},$$

where $z_i = z(p_i)$. Then we set

$$E(p, p_\infty) = \frac{E(z, 0)}{\sqrt{dz}}, \tag{28}$$

which is a constant multiple of

$$\frac{\theta[\varepsilon_o](I(p) \,|\, \Omega)}{\sqrt{h_{\varepsilon_o}(p)}}.$$

The proportional constant is determined so that the expansion of $E(p, p_\infty)\sqrt{dz}$ at p_∞ has the form $-z + O(z^2)$. So it depends on the choice of z. It is a section of the line bundle $L_\Delta^{-1} \otimes I^*\Theta$. The holomorphic line bundle $I^*\Theta$ on X is described by the transformation rule

$$\begin{aligned} f(p + \alpha_j) &= f(p), \\ f(p + \beta_j) &= e^{-\pi i \Omega_{jj} - 2\pi i (\int_{p_\infty}^p dv_j)} f(p). \end{aligned} \tag{29}$$

6.2 Theta function solution

Let

$$\omega(p_1, p_2) = d_{p_1} d_{p_2} \log E(p_1, p_2),$$

be the fundamental normalized differential of the second kind [12]. Using the local coordinate z we expand $\omega(p_1, p_2)$ and dv_i as as

$$\begin{aligned} \omega(p_1, p_2) &= \left(\frac{1}{(z_1 - z_2)^2} + \sum_{i,j=1}^{\infty} q_{ij} z_1^{i-1} z_2^{j-1} \right) dz_1 dz_2, \\ dv_i &= \left(\sum_{j=1}^{\infty} a_{ij} z^{j-1} \right) dz, \end{aligned} \tag{30}$$

where $z_i = z(p_i)$. Set

$$q(t) = \sum_{i,j=1}^{\infty} q_{ij} t_i t_j, \qquad A = (a_{ij})_{1 \le i \le g, 1 \le j}.$$

It is well known that

$$\tau(t) = e^{\frac{1}{2} q(t)} \theta(At + e \,|\, \Omega\,) \tag{31}$$

is a solution of the KP-hierarchy (16) for any $e = {}^t(e_1, ..., e_g) \in \mathbb{C}^g$ [19, 7, 35, 31, 15].

6.3 The point of UGM

Let us determine the point of UGM corresponding to (31). To this end we compute the adjoint wave function associated with $\tau(t)$.

Let dr_n, $n \geq 1$ be the meromorphic one form with a pole only at p_∞ of order $n+1$ which satisfies the following conditions:

$$dr_n = d\left(\frac{1}{z^n} + O(z)\right) \qquad \text{near } p_\infty,$$
$$\int_{\alpha_i} dr_n = 0 \text{ for any } i.$$

Then we have [19, 15, 25]

$$z^{-1}\Psi(t;z)\sqrt{dz} = \frac{1}{E(p,p_\infty)}\frac{\theta(I(p)+At+e)}{\theta(At+e)}\exp\Big(-\sum_{n=1}^{\infty} t_n \int^p dr_n\Big), \tag{32}$$

where the integral is normalized as

$$\lim_{p\to p_\infty}\left(\int^p dr_n - \frac{1}{z^n}\right) = 0.$$

Notice that

$$\theta(I(p)+At+e)\exp\Big(-\sum_{n=1}^{\infty} t_n \int^p dr_n\Big) \tag{33}$$

is a section of the holomorphic line bundle $I^*(\Theta)\otimes L_{-e}$ on X and specified by the transformation rule (34). Therefore $z^{-1}\Psi(t;z)\sqrt{dz}$ can be considered a section of the line bundle $L_\Delta \otimes L_{-e}$.

Now we define the map

$$\iota : H^0\left(X, (L_\Delta \otimes L_{-e})(*p_\infty)\right) \longrightarrow V,$$

in the following way.

First we specify the local trivialization of $L_\Delta \otimes L_{-e}$ as in (32). Namely a section of this bundle is written as $E(p,p_\infty)^{-1}$ times a section of $L_{-e}\otimes I^*\Theta$ and the latter is realized as a multiplicative functions on X which obeys the transformation rule given by

$$\begin{aligned} f(p+\alpha_j) &= f(p), \\ f(p+\beta_j) &= e^{-\pi i\Omega_{jj}-2\pi i(\int_{p_\infty}^p dv_j + e_j)} f(p). \end{aligned} \tag{34}$$

Take an element $\varphi(p)$ of $H^0\left(X, (L_\Delta \otimes L_{-e})(*p_\infty)\right)$ and expand it around p_∞ in z:

$$\varphi(p) = \sum_{-\infty<<n<\infty} c_n z^n \sqrt{dz}.$$

Then we set

$$\iota(\varphi) = \sum_{-\infty << n < \infty} c_n e_n \in V. \tag{35}$$

Define the subspace U of V as the image of ι:

$$U = \iota\left(H^0\left(X, (L_\Delta \otimes L_{-e})(*p_\infty)\right)\right). \tag{36}$$

Let λ be the partition defined by (7) if $\theta(e|\Omega) = 0$ and e is given by (5). We set $\lambda = \phi$ if $\theta(e|\Omega) \neq 0$. Then

Theorem 8 *The subspace U is a point of UGM^λ.*

Proof. For e satisfying $\theta(e|\Omega) \neq 0$ the theorem is proved in [15]. Let us prove the theorem in case $\theta(e|\Omega) = 0$.

Writing e as in (5) we have

$$\begin{aligned} L_\Delta \otimes L_{-e} &\simeq K_X(-q_1 - \cdots - q_{g-1}), & (37) \\ L_\Delta \otimes L_e &\simeq \mathcal{O}(q_1 + \cdots + q_{g-1}). & (38) \end{aligned}$$

Then

$$\begin{aligned} L_{-e+\delta} &= K_X(-q_1 - \cdots - q_{g-1} - (g-1)p_\infty), \\ L_{e+\delta} &= \mathcal{O}(q_1 + \cdots + q_{g-1} - (g-1)p_\infty). \end{aligned}$$

Therefore

$$\begin{aligned} (L_\Delta \otimes L_{-e})(np_\infty) &\simeq L_{-e+\delta}\left((g-1+n)p_\infty\right), & (39) \\ (L_\Delta \otimes L_e)(np_\infty) &\simeq L_{e+\delta}\left((g-1+n)p_\infty\right). & (40) \end{aligned}$$

Let

$$\begin{aligned} &0 \leq b'_1 < \cdots < b'_g \leq 2g-1, \\ &0 \leq b'^*_1 < b'^*_2 < \cdots, \end{aligned}$$

be gaps and non-gaps of $L_{-e+\delta}$ at p_∞ respectively.

By (39) we have

$$\dim(\mathrm{Ker}\,\pi_U) = \sharp\{i \,|\, b'^*_i - (g-1) \leq 0\}, \qquad \dim(\mathrm{Coker}\,\pi_U) = \sharp\{i \,|\, b'_i - (g-1) > 0\}.$$

Since the number of gaps is g we have

$$\dim(\mathrm{Coker}\,\pi_U) = g - \sharp\{i \,|\, b'_i - (g-1) \leq 0\}.$$

Then

$$\begin{aligned} & \dim(\operatorname{Ker}\pi_U)-\dim(\operatorname{Coker}\pi_U) \\ = \ & \sharp\{i\,|\,b_i^{\prime *}-(g-1)\le 0\}+\sharp\{i\,|\,b_i^{\prime}-(g-1)\le 0\}-g \\ = \ & g-g=0. \end{aligned}$$

Thus $U\in UGM$.

By the definition of gaps of $L_{-e+\delta}$ and (39) the sequence $\rho=(\rho(i))_{i<0}$ corresponding to U is given by

$$\rho(-i)=-b_i^{\prime *}+g-1. \tag{41}$$

In order to prove that U belongs to UGM^λ we need to rewrite this in terms of gaps of $L_{e+\delta}$.

Lemma 4 *(i)* $(b_1',...,b_g')=(2g-1-b_g^*,...,2g-1-b_1^*)$.
(ii) $(b_1^{\prime *},...,b_g^{\prime *})=(2g-1-b_g,...,2g-1-b_1)$.

Proof. For simplicity we denote $L_{\pm e+\delta}(kp_\infty)$ by $L_{\pm e+\delta}(k)$ respectively. By Riemann-Roch theorem, (39), (40) we have

$$h^0(L_{e+\delta}(g-1+n))-h^0(L_{-e+\delta}(g-1-n))=n. \tag{42}$$

This equation implies that $g-1+n$ is a gap of $L_{e+\delta}$ if and only if $g-n$ is a non-gap of $L_{-e+\delta}$. In fact the condition that $g-1+n$ is a gap of $L_{e+\delta}$ is equivalent to the equation

$$h^0(L_{e+\delta}(g-1+(n-1)))=h^0(L_{e+\delta}(g-1+n)),$$

which, by (42), is equivalent to

$$h^0(L_{-e+\delta}(g-n))=h^0(L_{-e+\delta}(g-n-1))+1.$$

The last equation is equivalent to the condition that $g-n$ is a non-gap of $L_{-e+\delta}$.

Let n be defined by $g-1+n=b_i$. Since b_i is a gap of $L_{e+\delta}$,

$$g-(b_i-g+1)=2g-1-b_i$$

is a non-gap of $L_{-e+\delta}$. Similarly $2g-1-b_i^*$ is a gap of $L_{-e+\delta}$ for any i. Since

$$0\le 2g-1-b_i,\,2g-1-b_i^*\le 2g-1, \qquad 1\le i\le g,$$

and these numbers are all distinct, they exhaust all gaps and non-gaps of $L_{-e+\delta}$ contained in $\{0,1,...,2g-1\}$. Thus the lemma is proved. $\Box$

Notice that $b_i^*=i+g-1$ for $i\ge g+1$. Then we have, by (2) of Lemma 4,

$$\begin{aligned} \rho &= (g-1-b_1^{\prime *},...,g-1-b_g^{\prime *},-(g+1),-(g+2),...) \\ &= (b_g-g,...,b_1-g,-(g+1),-(g+2),...). \end{aligned}$$

Thus

$$\lambda_\rho = \rho + (1,2,3,...) = (b_g - (g-1), ..., b_1, 0, 0, ...) = \lambda,$$

which completes the proof of Theorem 8. $\square$

Now we have

Theorem 9 *Let $\tau(t)$ be the solution of the KP-hierarchy given by (31) and U the point of UGM given by (36). Then*

$$C\tau(t) = \tau(t; U) \tag{43}$$

for some non-zero constant C.

Proof. Notice that the factors of automorphy of $f(p)$ in (34) do not depend on t. Thus if we expand (33) in $t = {}^t(t_1, t_2, ..)$, all the coefficients are sections of $I^*(\Theta) \otimes L_{-e}$. It implies that if we expand $z^{-1}\Psi(t;z)\sqrt{dz}\,\theta(At+e)$ in t, any coefficient is an element of $H^0(X, L_\Delta \otimes L_{-e}(*p_\infty))$.

Let m_0 be the order of the zero of $\tau(x, 0, ...)$ at $x = 0$. We expand $x^{m_0}\Psi(x, 0, ...; z)$ as in (26). Then

$$z^{-1}\Psi_i(z)\sqrt{dz} \in H^0(X, L_\Delta \otimes L_{-e}(*p_\infty)), \tag{44}$$

for any i. Let

$$\Psi_i(z) = \sum_{-\infty << n < \infty} \Psi_{in} z^n.$$

Then

$$\Psi_i(\partial^{-1})e_{-1} = \sum_{-\infty << n < \infty} \Psi_{in} e_{n-1}. \tag{45}$$

On the other hand we have, by the definition (35) of the map ι,

$$\iota(z^{-1}\Psi_i(z)\sqrt{dz}) = \iota(\sum_{-\infty << n < \infty} \Psi_{in} z^{n-1}\sqrt{dz}) = \sum_{-\infty << n < \infty} \Psi_{in} e_{n-1}. \tag{46}$$

Let

$$U' = \mathrm{Span}_{\mathbb{C}}\{\Psi(\partial^{-1})e_{-1} \,|\, i \geq 0\}.$$

Then U' is the point of UGM corresponding to the solution (31) by Proposition 3 and Theorem 7. By (44), (45) and (46) we have $U' \subset U$. Since a strict inclusion relation is impossible for two points of UGM we have $U' = U$. Thus the theorem is proved. $\square$

As a corollary of Theorem 9 we have, by (24),

Corollary 1 *The $\tau(t)$ given by (31) has the expansion of the form*

$$C\tau(t) = s_\lambda(t) + \sum_{\lambda < \mu} \xi_\mu s_\mu(t),$$

for some constant C.

7 Series expansion of the theta function

In this section we assume that a data $(X, \{\alpha_i, \beta_i\}, p_\infty, z, e)$ is given, where notation is as before.

7.1 Taylor expansion

In order to study the series expansion of the theta function we need to change the variables to appropriate ones.

Let $\{du_{w_i}\}$ be a basis of holomorphic one forms such that du_{w_i} has the following expansion at p_∞:

$$du_{w_i} = z^{w_i - 1}(1 + O(z))dz. \tag{47}$$

The existence of such a basis is easily proved using Riemann-Roch theorem (c.f. [18]). However a basis which satisfies the condition (47) is not unique. For the moment we do not know what is the best choice. So we take any one of them.

By integrating this basis over the canonical homology basis we define the period matrices ω_1, ω_2 by

$$2\omega_1 = \left(\int_{\alpha_j} du_i \right), \qquad 2\omega_2 = \left(\int_{\beta_j} du_i \right).$$

The normalized period matrix Ω is given by $\Omega = \omega_1^{-1}\omega_2$.

We label the g variables of the theta function by gaps at p_∞ as

$$u = {}^t(u_{w_1}, ..., u_{w_g}),$$

and consider the function of the form

$$\theta\big((2\omega_1)^{-1}u + e \,|\, \Omega\big).$$

We set $\partial_i = \partial/\partial u_i$ and $\partial_I = \partial_{i_1} \cdots \partial_{i_r}$ for $I = (i_1, ..., i_r)$. If we differentiate theta function i_j should be considered in $\{w_1, ..., w_g\}$. We assign weight i to u_i. Then the initial term, with respect to weight, of the Taylor series expansion of this function is given by the following theorem.

Theorem 10 *Suppose that a data $(X, \{\alpha_i, \beta_i\}, p_\infty, z, \{du_{w_i}\}, e)$ is given. Let λ be the partition associated with it. Then we have*

$$C\theta\big((2\omega_1)^{-1}u + e \,|\, \Omega\big) = s_\lambda(t)|_{t_{w_i} = u_{w_i}} + \textit{higher weight terms}.$$

where $C = c_0 \partial_{A_0} \theta(e|\Omega)$ and $c_0 = \pm 1$ is given by (10).

Proof. Let us expand du_{w_i} at p_∞ in z as

$$du_{w_i} = \sum_{j=1}^{\infty} b_{ij} z^{j-1} dz,$$

and define the matrix $B = (b_{ij})$. By (47) B has the following triangular structure:

$$b_{ij} = \begin{cases} 0 & \text{if } j < w_i \\ 1 & \text{if } j = w_i. \end{cases} \tag{48}$$

Lemma 5 *We have $B = 2\omega_1 A$.*

Proof. Write

$$du_{w_i} = \sum_{j=1}^{g} c_{ij} dv_j. \tag{49}$$

Integrating this equation over α_j we get $(2\omega_1)_{ij} = c_{ij}$. Then the lemma follows by comparing the expansion coefficients of (49). □

By the lemma the tau function (31) can be written as

$$\tau(t) = e^{\frac{1}{2}q(t)} \theta((2\omega_1)^{-1} Bt + e \,|\, \Omega). \tag{50}$$

Let $u = {}^t(u_{w_1}, ..., u_{w_g}) = Bt$ and set $t_j = 0$ for all j except t_{w_i}, $1 \le i \le g$. Then we can write

$$u = \tilde{B}\tilde{t}, \qquad \tilde{t} = {}^t(t_{w_1}, ..., t_{w_g}),$$

where the $g \times g$ matrix $\tilde{B} = (\tilde{b}_{ij})$ is the upper triangular matrix whose diagonal entries are all equal to 1 due to (48). Thus $\tilde{B}$ is invertible and $\tilde{B}^{-1}$ has a similar form to $\tilde{B}$. Then

$$\begin{aligned} &\theta((2\omega_1)^{-1}u + e \,|\, \Omega) = e^{-\frac{1}{2}q(\tilde{u})} \tau(\tilde{u}), \\ &\tilde{u} = {}^t(\tilde{u}_{w_1}, ..., \tilde{u}_{w_g}) = \tilde{B}^{-1} u \end{aligned} \tag{51}$$

where $\tilde{u}$ in $\tau(t)$and $q(t)$ should be understood that $\tilde{u}_{w_i}$ sits on the w_i-th component of $t = {}^t(t_1, t_2, ...)$ and other components of t are zero. Since $\tilde{u}_{w_i} = u_{w_i}$ modulo higher weight terms, the theorem follows from Corollary 1 and Theorem 2 (iii). □

7.2 Duality

There is a relation of the expansions of the theta function at e and at $-e$.

Theorem 11 *Under the same assumption as in Theorem 10 we have*

$$(-1)^{|\lambda|} C\theta\big((2\omega_1)^{-1}u - e \,|\, \Omega\big) = s_{{}^t\lambda}(t)|_{t_{w_i} = u_{w_i}} + \textit{higher weight terms},$$

where ${}^t\lambda$ is the conjugate partition of λ and C is the same as in Theorem 10.

Proof. Since $\theta(Z\,|\,\Omega)$ is an even function of Z

$$\theta\big((2\omega_1)^{-1}u - e\,|\,\Omega\big) = \theta\big((2\omega_1)^{-1}(-u) + e\,|\,\Omega\big).$$

Then the theorem follows from the known relation for the Schur functions [20]

$$s_\mu(-t) = (-1)^{|\mu|} s_{{}^t\mu}(t),$$

and Theorem 10. □

Corollary 2 *Let λ be the partition associated with $(X, \{\alpha_i, \beta_i\}, p_\infty, e)$. Then the partition associated with $(X, \{\alpha_i, \beta_i\}, p_\infty, -e)$ is ${}^t\lambda$.*

Proof. Since $s_\mu(t) = s_\nu(t)$ implies $\mu = \nu$ for two partitions μ and ν, the assertion follows from Theorem 10 and Theorem 11. □

7.3 Examples

We give examples of the partition λ given by (7) which, by Theorem 10, determines the initial term of the Taylor expansion of the theta function.

Example 1 Let X be any compact Riemann surface of genus g , p_∞ a non-Weierstrass point of X and $e = -\delta$. In this case $L_{e+\delta} \simeq \mathcal{O}$ and we have

$$(b_1, ..., b_g) = (w_1, ..., w_g) = (1, 2, ..., g).$$

Thus $\lambda = (1^g)$. In this case $s_\lambda(t) = (-1)^g p_g(-t)$. For example, for $1 \le g \le 4$ they are given by

$$s_{(1)}(t) = t_1, \qquad s_{(1,1)}(t) = -t_2 + \frac{t_1^2}{2}, \qquad s_{(1,1,1)}(t) = t_3 - t_1 t_2 + \frac{t_1^3}{3!},$$
$$s_{(1,1,1,1)}(t) = -t_4 + t_1 t_3 + \frac{t_2^2}{2} - \frac{t_1^2 t_2}{2} + \frac{t_1^4}{4!}.$$

The $(3,4,5)$ curve studied in [16] is included in this Example.

Example 2 Let X and p_∞ be the same as in Example 1. Take $e = \delta$. Then, by duality, $\lambda = (g)$. In this case $s_\lambda(t) = p_g(t)$.

Example 3 Let X be an (n, s) curve [5] or a telescopic curve of genus g [21][1][2]. Take $p_\infty = \infty$ and $e = -\delta$. Then

$$\lambda = (w_g, ..., w_1) - (g-1, ..., 1, 0). \tag{52}$$

In this case $\delta = -\delta$ in $J(X)$ and λ satisfies ${}^t\lambda = \lambda$ by Corollary 2. These are cases which were studied in most of the literatures.

Example 4 In [17] sigma functions of $(3,7,8)$ $(g=4)$ and $(6,13,14,15,16)$ $(g=12)$ curves are studied. If we take $e=-\delta$, then (52) holds. For $(3,7,8)$-curve, gaps are (1,2,4,5) and $\lambda=(2,2,1,1)$. For $(6,13,14,15,16)$-curve, gaps are (1,2,3,4,5,7,8,9,10,11,17,23) and $\lambda=(12,7,2,2,2,2,2,1,1,1,1,1)$. In the latter case the maximal gap 23 is equal to $2g-1$. It implies $\delta=-\delta$ in $J(X)$ and $\lambda={}^t\lambda$.

If we take $e=\delta$, then we have $\lambda=(4,2)$ for $(3,7,8)$-curve by duality.

7.4 The case of hyperelliptic curves

In the case of hyperelliptic curves it is possible to determine the partition λ corresponding to any point on the theta divisor explicitly.

Let X be the hyperelliptic curve of genus g given by

$$y^2=f(x), \qquad f(x)=\prod_{i=1}^{2g+1}(x-e_i),$$

where $e_i\neq e_i$ for any $i\neq j$. We take a canonical homology basis $\{\alpha_i,\beta_i\}$ and the base point $p_\infty=\infty$. Let ϕ be the hyperelliptic involution, $\phi(x,y)=(x,-y)$. It is known that the points on the theta divisor with the multiplicity $m_0\geq 1$ are given by

$$q_1+\cdots+q_{g+1-2m_0}+(2m_0-2)\infty-\Delta, \tag{53}$$

where $q_i's$ are points on X satisfying $q_i\neq\phi(q_j)$ for any $i\neq j$ [12]. We denote this set of points by Θ_{m_0}.

We set

$$\begin{aligned}\Theta_{m_0,o} &= \{q_1+\cdots+q_{g+1-2m_0}+(2m_0-2)\infty-\Delta\in\Theta_{m_0}\,|\,q_i\neq\infty\ \forall i\},\\ \Theta_{m_0,e} &= \{q_1+\cdots+q_{g-2m_0}+(2m_0-1)\infty-\Delta\in\Theta_{m_0}\,|\,q_i\neq\infty\ \forall i\}.\end{aligned}$$

Then

Proposition 4 *The partition λ associated with $(X,\{\alpha_i,\beta_i\},\infty,e)$ is given as follows.*

(i) $\lambda=(2m_0-1,2m_0-2,...,1)$ for $e\in\Theta_{m_0,o}$.

(ii) $\lambda=(2m_0,2m_0-1,...,1)$ for $e\in\Theta_{m_0,e}$.

This proposition follows from

Lemma 6 *The gaps of $L_{e+\delta}$ are given as follows.*

(i) $\{0,1,...,g-2m_0,g-2m_0+2i\ (1\leq i\leq 2m_0-1)\}$ for $e\in\Theta_{m_0,o}$,

(ii) $\{0,1,...,g-1-2m_0,g-1-2m_0+2i\ (1\leq i\leq 2m_0)\}$ for $e\in\Theta_{m_0,e}$.

Proof. Let us prove (i). The proof of (ii) is similar. We have

$$e + \delta = q_1 + \cdots + q_{g+1-2m_0} - (g + 1 - 2m_0)\infty.$$

Since

$$L_{e+\delta}(n\infty) \simeq \mathcal{O}_X\big(q_1 + \cdots + q_{g+1-2m_0} + (n - (g + 1 - 2m_0))\infty\big),$$

$H^0(X, L_{e+\delta}(n\infty))$ is identified with the space of meromorphic functions on X with at most a simple pole at q_i ($1 \le i \le g+1-2m_0$) and a pole of order at most $n-(g+1-2m_0)$ at ∞ if $n - (g + 1 - 2m_0) \ge 0$ or with at most a simple pole at q_i ($1 \le i \le g + 1 - 2m_0$) and a zero of order at least $-n + (g + 1 - 2m_0)$ at ∞ if $n - (g + 1 - 2m_0) < 0$.

Let $q_i = (x_i, y_i)$. Since a meromorphic function on X which has a pole only at ∞ is a polynomial of x and y, any element of $H^0(X, L_{e+\delta}(n\infty))$ can be written in the form

$$F(x, y) = \frac{A(x) + B(x)y}{(x - x_1)\cdots(x - x_{g+1-2m_0})}, \tag{54}$$

$A(x)$ and $B(x)$ are some polynomials of x. Since F has at most a simple pole at $q_i = (x_i, y_i)$, $A(x)$ and $B(x)$ should satisfy

$$A(x_i) - B(x_i)y_i = 0, \quad 1 \le i \le g + 1 - 2m_0. \tag{55}$$

First consider the case $n < g + 1 - 2m_0$. In this case F must have a zero at ∞ of order at least $g + 1 - 2m_0 - n$. Let us estimate the order of zero or pole at ∞ of F given by (54).

If $B(x) = 0$, then by (55), $A(x)$ must be divided by $\prod_{i=1}^{g+1-2m_0}(x-x_i)$ and F becomes a polynomial of x. Thus F can not have a zero at ∞.

Suppose that $B(x) \neq 0$. Let k and l be degrees of $A(x)$ and $B(x)$ respectively. Since x and y has a pole of order 2 and $2g + 1$ at ∞, the order of a zero of F at ∞, denoted by $\mathrm{ord}(F)$, is given by

$$\mathrm{ord}(F) = 2(g + 1 - 2m_0) - \max\{2k, 2l + 2g + 1\}.$$

Notice that $\max\{2k, 2l + 2g + 1\} \ge 2g + 1$. Then

$$\mathrm{ord}(F) \le 2(g + 1 - 2m_0) - (2g + 1) = 1 - 4m_0 < 0.$$

Thus F can not have a zero at ∞ in this case too. Consequently $0, 1, ..., g - 2m_0$ are gaps of $L_{e+\delta}$ at ∞.

If $n = g + 1 - 2m_0$, the constant function 1 can be considered as a global section of $L_{e+\delta}(n\infty)$. Thus n is a non-gap.

Suppose that $n > g + 1 - 2m_0$. In this case x^{2i}, $i \ge 1$ gives a global section of $L_{e+\delta}(n\infty)$ with $n = 2i + (g + 1 - 2m_0)$. Let us prove that $n = 2i - 1 + (g + 1 - 2m_0)$, $1 \le i \le 2m_0 - 1$ is a gap. The order of a pole of F at ∞, which is denoted by $-\mathrm{ord}(F)$, is given by

$$-\mathrm{ord}(F) = \max\{2k, 2l + 2g + 1\} - 2(g + 1 - 2m_0) \ge 2l - 1 + 4m_0.$$

Therefore, if $n = 2i - 1 + (g + 1 - 2m_0)$ is a non gap, it must satisfy

$$2i - 1 \geq 2l - 1 + 4m_0,$$

which implies $i \geq 2m_0 + l \geq 2m_0$. Thus $n = 2i - 1 + (g + 1 - 2m_0)$, $1 \leq i \leq 2m_0 - 1$ is a gap.

The number of gaps we obtained so far is $(g - 2m_0 + 1) + (2m_0 - 1) = g$. Thus those exhaust all gaps. Thereby the proof of the proposition is completed. $\square$

Remark If g is odd and $m_0 = \frac{g+1}{2}$ then $e = -\delta \in \Theta_{m_0,o}$. In this case $\lambda = (g, g-1, ..., 1)$. If g is even and $m_0 = \frac{g}{2}$, then $e = -\delta \in \Theta_{m_0,e}$ and $\lambda = (g, g-1, ..., 1)$. These results are the same as those given in Example 3 for $(2, 2g+1)$ curves as it should be.

7.5 Expansion on Abel-Jacobi images

We denote by $l(\mu)$ the length of a partition μ.

Proposition 5 *Let $0 \leq k \leq l(\lambda)$. Then, for any $I = (i_1, ..., i_m)$, $m \geq 1$, satisfying $\sum_{j=1}^m i_j < N_{\lambda,k}$ we have*

$$\partial_I \theta(\sum_{i=1}^k (p_i - p_\infty) + e \,|\, \Omega) = 0,$$

for any $p_i \in X$, $1 \leq i \leq k$.

Proof. Differentiate (51) by ∂_I and use Proposition 4 in [27], Corollary 1, to get the result. $\square$

Theorem 12 *Suppose that $m_k > 0$. Let c_k be given by (10) and $z_j = z(p_j)$ for $p_j \in X$. Then*

(i) $C\partial_{A_k} \theta(\sum_{j=1}^k (p_j - p_\infty) + e \,|\, \Omega) = c_k s_{(\lambda_1,...,\lambda_k)}(\sum_{j=1}^k [z_j]) +$ *higher weight terms,*

where C is the constant in Theorem 10 and $c_k = \pm 1$ is given by (10).

(ii) $\partial_{A_k} \theta(\sum_{j=1}^k (p_j - p_\infty) + e \,|\, \Omega)$

$$= \frac{c_k}{c_{k-1}} \partial_{A_{k-1}} \theta(\sum_{j=1}^{k-1} (p_j - p_\infty) + e \,|\, \Omega) z_k^{\lambda_k} + O(z_k^{\lambda_k + 1}),$$

where $z_j = z(p_j)$.

Proof. The theorem easilly follows from Corollary 1, Theorem 3 and (50) as in the proof of Corollary 4 in [27]. $\square$

7.6 Refined Riemann's singularity theorem

Corollary 3 *We assume the same conditions as in Theorem 10. Then*
(i) For any $I = (i_1, ..., i_m)$*,* $m \geq 1$*, satisfying* $\sum_{j=1}^{m} i_j < |\lambda|$ *we have*

$$\partial_I \theta(e \,|\, \Omega) = 0.$$

(ii) If $m < m_0$ *we have*

$$\partial_I \theta(e \,|\, \Omega) = 0,$$

for any $I = (i_1, ..., i_m)$*.*
(iii) $\partial_{A_0} \theta(e \,|\, \Omega) \neq 0$*.*

Proof. (i) is the special case $k = 0$ of Proposition 5. (ii) follows from Theorem 4 and (51). (iii) is the special case $k = 0$ of Theorem 12 (i). □

Notice that Riemann's singularity theorem is a part of this corollary. Moreover the assertion (iii) gives one non-vanishing derivative with degree m_0 explicitly and (i) gives a new vanishing property of the theta function which does not follow from Riemann's singularity theorem. In this sense Corollary 3 is an extension and a refinement of Riemann's singularity theorem. By Proposition 2, we can say more.

Corollary 4 *In the series expansion of the theta function* $C\theta((2\omega_1)^{-1}u + e \,|\Omega)$ *at* $u = 0$ *the terms with the minimal weight and the minimal degree are given by* $L_\lambda(t)|_{t_{w_i} = u_{w_i}}$*, where* C *is the constant in Theorem 10.*

8 Sigma function

Sigma functions of an arbitrary Riemann surface have been introduced by Korotkin and Shramchenko[18]. In this section we first review their construction in a more unified frame work. Next we prove the modular invariance of sigma functions with characteristics by specifying a suitable normalization constant which is obtained as a result of Corollary 3 and is apparently different from that in [18].

In this section we assume that a data $(X, \{\alpha_i, \beta_i\}, p_\infty, z, e, \{du_{w_i}\})$ is given.

8.1 Bilinear meromorphic differential

The key ingredients in constructing sigma function are certain bilinear meromorphic differentials. So we first study general properties of such bilinear differentials.

Let $\widehat{\omega}(p_1, p_2)$ be a symmetric bilinear meromorphic differential on $X \times X$ such that it is holomorphic outside the diagonal $\{(p, p) | \, p \in X\}$ where it has a double pole and at any $p_0 \in X$ it has the expansion of the form

$$\widehat{\omega}(p_1, p_2) = \left(\frac{1}{(w_1 - w_2)^2} + \text{holomorphic in } w_1, w_2 \right) dw_1 dw_2,$$

where w is a local coordinate at p_0 and $w_i = w(p_i)$.

The fundamental normalized differential of the second kind $\omega(p_1, p_2)$ is an example (see (30)). In general $\widehat{\omega}(p_1, p_2)$ has the following structure.

Proposition 6 *For any $\widehat{\omega}(p_1, p_2)$ there exist $\Omega(p_1, p_2)$ and $\{d\tilde{r}_i\}_{i=1}^{\infty}$ such that*

$$\widehat{\omega}(p_1, p_2) = d_{p_2}\widehat{\Omega}(p_1, p_2) + \sum_{i=1}^{g} du_{w_i}(p_1) d\tilde{r}_i(p_2). \tag{56}$$

Here $d\tilde{r}_i(p)$ is a locally exact meromorphic one form on X which has a pole only at p_∞ and $\widehat{\Omega}(p_1, p_2)$ is a meromorphic one form on $X \times X$ which satisfies the following conditions.

(i) It is a meromorphic one form in p_1 for a fixed p_2.

(ii) It is a meromorphic function in p_2 for a fixed p_1.

(iii) It is holomorphic except $\{(p, p) | \, p \in X\} \cup \{(p_\infty, p) | \, p \in X\} \cup \{(p, p_\infty) | \, p \in X\}$.

(iv) It has a simple pole at the diagonal $\{(p, p) | \, p \in X\}$.

Proof. By Lemma 7 of [24] $\widehat{\omega}$ can be written in terms of ω as

$$\widehat{\omega}(p_1, p_2) = \omega(p_1, p_2) + \sum_{i,j=1}^{g} c_{ij} du_{w_i}(p_1) du_{w_j}(p_2),$$

for some constants c_{ij} satisfying $c_{ij} = c_{ji}$. Therefore it is sufficient to prove (56) for ω.

Let us consider the exact sequence of sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}(*p_\infty) \longrightarrow d\mathcal{O}(*p_\infty) \longrightarrow 0,$$

where $d : \mathcal{O}(*p_\infty) \to K_X(*p_\infty)$ is the exterior differentiation. Taking cohomologies we have

$$H^1(X, \mathbb{C}) \simeq H^0(X, d\mathcal{O}(*p_\infty))/dH^0(X, \mathcal{O}(*p_\infty)). \tag{57}$$

We remark that $H^0(X, d\mathcal{O}(*p_\infty))$ is the space of locally exact meromorphic one forms with a pole only at p_∞. Thus this space is generated, as a vector space, by holomorphic one forms and the differentials of the second kind with a pole only at p_∞. We realize $H^1(X, \mathbb{C})$ as in the right hand side of (57).

Let $\{du_i^{(1)}, du_i^{(2)}\}$ be the basis of $H^1(X, \mathbb{C})$ which is dual to $\{\alpha_i, \beta_i\}$. Namely they satisfy

$$\int_{\alpha_j} du_i^{(1)} = \delta_{ij}, \qquad \int_{\alpha_j} du_i^{(2)} = 0,$$
$$\int_{\beta_j} du_i^{(1)} = 0, \qquad \int_{\beta_j} du_i^{(2)} = \delta_{ij}. \tag{58}$$

We set

$$\widehat{\Omega}(p_1,p_2)=d_{p_1}\log\frac{E(p_1,p_2)}{E(p_1,p_\infty)}-2\pi i\sum_{k=1}^{g}dv_k(p_1)\int^{p_2}du_k^{(2)}. \tag{59}$$

We show that this $\widehat{\Omega}(p_1,p_2)$ satisfies properties (i) - (iv).

Since the first term in the right hand side is invariant when p_1 goes round α_i or β_i cycles due to the transformation rule of the theta function, (i) is obvious. Since $E(p_1,p_2)$ has a unique simple zero at $p_1=p_2$, (iii) and (iv) are also obvious. Let us prove (ii). We have to show that $\widehat{\Omega}(p_1,p_2)$ is invariant when p_2 goes round α_i or β_i.

If p_2 goes round α_j then $\widehat{\Omega}(p_1,p_2)$ is invariant due to the transformation rule of the theta function and (58). Next consider the case where p_2 goes round β_j. Then the integral of the normalized holomorphic differential changes as

$$\int_{p_1}^{p_2}dv\mapsto\int_{p_1}^{p_2}dv+\Omega\mathbf{e}_j,$$

where $\mathbf{e}_j={}^t(0,...,1,...,0)$ is the j-th unit vector and Ω is the normalized period matrix. Since

$$\begin{aligned}&\theta[\varepsilon_o](\int_{p_1}^{p_2}dv+\Omega\mathbf{e}_j)\\&=e^{-2\pi i(\varepsilon_o'')_j-\pi i\Omega_{jj}-2\pi i\int_{p_1}^{p_2}dv_j}\theta[\varepsilon_o](\int_{p_1}^{p_2}dv),\end{aligned}$$

we have

$$d_{p_1}\log\frac{E(p_1,p_2)}{E(p_1,p_\infty)}\mapsto d_{p_1}\log\frac{E(p_1,p_2)}{E(p_1,p_\infty)}+2\pi i dv_j(p_1). \tag{60}$$

On the other hand we have

$$\int^{p_2}du_k^{(2)}\mapsto\int^{p_2}du_k^{(2)}+\delta_{jk}.$$

Thus

$$-2\pi i\sum_{k=1}^{g}dv_k(p_1)\int^{p_2}du_k^{(2)}\mapsto-2\pi i\sum_{k=1}^{g}dv_k(p_1)\int^{p_2}du_k^{(2)}-2\pi i dv_j(p_1). \tag{61}$$

By (60) and (61) $\widehat{\Omega}(p_1,p_2)$ is invariant when p_2 goes round β_j. Thus the property (ii) is proved.

Next let us prove (56) by defining $d\tilde{r}_i$ explicitly. Taking d_{p_2} of (59) we get

$$d_{p_2}\widehat{\Omega}(p_1,p_2)=\omega(p_1,p_2)-2\pi i\sum_{j=1}^{g}dv_j(p_1)du_j^{(2)}(p_2). \tag{62}$$

Using $dv_i = \sum_{j=1}^{g} c_{ij} du_{w_j}$, $c_{ij} = ((2\omega_1)^{-1})_{ij}$, we have

$$2\pi i \sum_{j=1}^{g} dv_j(p_1) du_j^{(2)}(p_2) = \sum_{j=1}^{g} du_{w_j}(p_1) d\tilde{r}_j(p_2), \qquad d\tilde{r}_j := 2\pi i \sum_{k=1}^{g} c_{kj} du_k^{(2)}.$$

Since $d\tilde{r}_j$ is a locally exact meromorphic one form with a pole only at p_∞ by construction, the relation (56) is proved. □

Corollary 5 *$\{du_{w_i}, d\tilde{r}_i\}$ is a symplectic basis of $H^1(X, \mathbb{C})$ with respect to the intersection form $\bullet$:*

$$du_{w_i} \bullet du_{w_j} = d\tilde{r}_i \bullet d\tilde{r}_j = 0, \qquad du_{w_i} \bullet d\tilde{r}_j = \delta_{ij}. \tag{63}$$

Proof. The computation of the intersection is the same as that in Proposition 3 in [24] using Proposition 6. □

Let us define the period matrices η_i, $i = 1, 2$ by

$$-2\eta_1 = \left(\int_{\alpha_j} d\tilde{r}_i \right), \qquad -2\eta_2 = \left(\int_{\beta_j} d\tilde{r}_i \right),$$

and set

$$M = \begin{pmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{pmatrix}.$$

As a consequence of (63) M satisfies the Riemann bilinear relation of the following form [24]

$${}^t M \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M = -\frac{\pi i}{2} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

It guarantees the consistency of the transformation rule of sigma functions [4].

The following corollary shows that we can determine η_1, η_2 directly from $\widehat{\omega}(p_1, p_2)$.

Corollary 6 *The following relations are valid:*

$$\int_{\alpha_j} \widehat{\omega}(p_1, p_2) = \sum_{i=1}^{g} du_{w_i}(p_1)(-2\eta_{1,ij}), \tag{64}$$

$$\int_{\beta_j} \widehat{\omega}(p_1, p_2) = \sum_{i=1}^{g} du_{w_i}(p_1)(-2\eta_{2,ij}), \tag{65}$$

where the integrals are taken with respect to the variable p_2.

8.2 Klein form

In defining sigma functions we need a meromorphic bilinear differential $\widehat{\omega}(p_1, p_2)$ which is independent of the choice of canonical homology basis. The Klein differential [14, 12] is one of such differentials. Let us recall its definition.

Let $\mathcal{S}$ be the set of half characteristics $\varepsilon = {}^t(\varepsilon', \varepsilon'')$, $\varepsilon', \varepsilon'' \in (\frac{1}{2}\mathbb{Z}^g)/\mathbb{Z}^g$ such that $\theta[\varepsilon](0|\Omega)$ does not vanish and $N(\mathcal{S})$ denote the number of elements of $\mathcal{S}$. Then the Klein form $\omega_K(p_1, p_2)$ is defined by

$$\omega_K(p_1, p_2) = \omega(p_1, p_2) + \frac{1}{N(\mathcal{S})} \sum_{i,j=1}^{g} dv_i(p_1) dv_j(p_2) \frac{\partial^2}{\partial z_i \partial z_j} \log \mathcal{F}(z)|_{z=0},$$

$$\mathcal{F}(z) = \prod_{\varepsilon \in \mathcal{S}} \theta[\varepsilon](z|\Omega).$$

Using the transformation formula of the theta function under the action of the symplectic group, it can be proved that ω_K is independent of the choice of canonical homology basis [12, 18, 14].

We can easily compute periods of Klein form $\omega_K(p_1, p_2)$ using

$$\int_{\alpha_j} \omega(p_1, p_2) = 0, \qquad \int_{\beta_j} \omega(p_1, p_2) = 2\pi i dv_j(p_1),$$

where integrals are with respect to the variable p_2. The result is given by (64) and (65) with

$$\eta_1 = {}^t\omega_1^{-1}\Lambda, \qquad \eta_2 = -\frac{\pi}{2} {}^t\omega_1^{-1} + {}^t\omega_1^{-1}\Lambda\Omega,$$

where $\Lambda = (\Lambda_{ij})$ is given by

$$\Lambda_{ij} = -\frac{1}{4N(S)} \frac{\partial^2}{\partial z_i \partial z_j} \log \mathcal{F}(z)|_{z=0}.$$

These η_i's are nothing but the ones given in [18].

8.3 Definition of sigma function

Take any $\widehat{\omega}(p_1, p_2)$ of the form

$$\widehat{\omega}(p_1, p_2) = \omega_K(p_1, p_2) + \sum_{i,j=1}^{g} c_{ij} du_{w_i}(p_1) du_{w_j}(p_2), \tag{66}$$

where $c_{ij} = c_{ji}$ is a constant independent of the choice of canonical homology basis. Define the period matrices η_i by (64) and (65). Let $\varepsilon = {}^t(\varepsilon', \varepsilon'')$, $\varepsilon', \varepsilon'' \in \mathbb{R}^g$ be the characteristics of e:

$$e = q_1 + \cdots + q_{g-1} - \Delta = \Omega\varepsilon' + \varepsilon'',$$

and $u = {}^t(u_{w_1}, ..., u_{w_g})$. Notice that if we change the choice of canonical homology basis $\{\alpha_i, \beta_i\}$, then Δ, Ω and the Abel-Jacobi map are changed. Consequently ε also depends on the choice of canonical homology basis.

By Corollary 3 we know

$$\partial_{A_0}\theta[\varepsilon](0 \,|\, \Omega) \neq 0.$$

Taking this quantity as a normalization constant we define the sigma function as follows.

Definition 9 *The sigma function associated with $(X, \{\alpha_i, \beta_i\}, p_\infty, z, e, \{du_{w_i}\}, \widehat{\omega})$ is defined by*

$$\sigma[\varepsilon](u) = \exp(\frac{1}{2}{}^t u \eta_1 \omega_1^{-1} u) \frac{\theta[\varepsilon]((2\omega_1)^{-1}u \,|\, \Omega)}{c_0 \partial_{A_0}\theta[\varepsilon](0 \,|\, \Omega)}, \tag{67}$$

where $c_0 = \pm 1$ is given by (10).

8.4 Modular invariance

Theorem 13 *The function $\sigma[\varepsilon](u)$ is independent of the choice of $\{\alpha_i, \beta_i\}$ and has the series expansion of the form*

$$\sigma[\varepsilon](u) = s_\lambda(t)|_{t_{w_i} = u_{w_i}} + \textit{higher weight terms}, \tag{68}$$

where λ is given by (7).

Proof. The series expansion (68) follows from Theorem 10. The invariance under the change of the canonical homology basis is proved based on the following transformation formula for the theta function in [13].

Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an element of the symplectic group $Sp(2g, \mathbb{Z})$, where A, B, C, D are integral matrices of degree g, and $z = {}^t(z_1, ..., z_g)$. We set

$$\tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1}, \qquad \tilde{z} = {}^t(C\Omega + D)^{-1} z, \tag{69}$$

$$\tilde{\varepsilon} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \varepsilon + \frac{1}{2}\mathrm{diag}\begin{pmatrix} C^tD \\ A^tB \end{pmatrix}, \tag{70}$$

where $\mathrm{diag}(\cdot)$ denotes the column vector whose components are the diagonal entries of the matrices in $(\cdot)$. Then

$$\theta[\tilde{\varepsilon}](\tilde{z} \,|\, \tilde{\Omega}) = \gamma(\det(C\Omega + D))^{\frac{1}{2}} e^{\pi i {}^t z (C\Omega + D)^{-1} C z} \theta[\varepsilon](\, z \,|\, \Omega), \tag{71}$$

where γ is some 8-th root of unity.

Let $\alpha = {}^t(\alpha_1, ..., \alpha_g)$, $\beta = {}^t(\beta_1, ..., \beta_g)$. Let

$$T = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

be an element of $Sp(2g, \mathbb{Z})$. We change the canonical homology basis by T:

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = T \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{72}$$

Notice that T is an element of $Sp(2g, \mathbb{Z})$ if and only if M is an element of $Sp(2g, \mathbb{Z})$. Let $\tilde{\omega}_i, \tilde{\eta}_i$ be the period matrices corresponding to $\{\tilde{\alpha}_i, \tilde{\beta}_i\}$. Since the bilinear differential $\widehat{\omega}(p_1, p_2)$ does not depend on the choice of a canonical homology basis we have, by computation,

$$\tilde{\omega}_1 = \omega_1{}^t D + \omega_2{}^t C, \qquad \tilde{\omega}_2 = \omega_1{}^t B + \omega_2{}^t A,$$
$$\tilde{\eta}_1 = \eta_1{}^t D + \eta_2{}^t C, \qquad \tilde{\eta}_2 = \eta_1{}^t B + \eta_2{}^t A.$$

Then the normalized period matrix $\tilde{\Omega}$ corresponding to $\{\tilde{\alpha}_i, \tilde{\beta}_i\}$ becomes

$$\tilde{\Omega} = {}^t\tilde{\Omega} = {}^t(\tilde{\omega}_1^{-1}\tilde{\omega}_2) = (A\Omega + B)(C\Omega + D)^{-1}.$$

Using Riemann's vanishing theorem we see that Riemann's constant δ changes to (see also [12])

$$\tilde{\delta} = {}^t(C\Omega + D)^{-1}\delta - \tilde{\Omega}\zeta' - \zeta'',$$

where $\zeta', \zeta'' \in (1/2\mathbb{Z}^g)/\mathbb{Z}^g$ are given by

$$\begin{pmatrix} \zeta' \\ \zeta'' \end{pmatrix} = \frac{1}{2}\mathrm{diag}\begin{pmatrix} C{}^tD \\ A{}^tB \end{pmatrix}.$$

Then e transforms to $\tilde{e} = \Omega\tilde{\varepsilon}' + \tilde{\varepsilon}''$ with ε being given by (70).

Notice that

$$(2\tilde{\omega}_1)^{-1} = {}^t(C\Omega + D)^{-1}(2\omega_1)^{-1}.$$

Therefore if we change the canonical homology basis to $\{\tilde{\alpha}_i, \tilde{\beta}_i\}$ by (72), the theta function $\theta[\varepsilon]((2\omega_1)^{-1}u\,|\Omega)$ changes to

$$\theta[\tilde{\varepsilon}]({}^t(C\Omega + D)^{-1}(2\omega_1)^{-1}u\,|\tilde{\Omega}),$$

where $\tilde{\Omega}$, $\tilde{\varepsilon}$ are given by (69), (70). Thus the formula (71) can be applied. Then we have

$$\begin{aligned} \theta[\tilde{\varepsilon}]((2\tilde{\omega}_1)^{-1}u\,|\tilde{\Omega}) &= \gamma\,(\det(C\Omega + D))^{1/2}\, e^{\pi i {}^tu {}^t(2\omega_1)^{-1}(C\Omega+D)^{-1}C(2\omega_1)^{-1}u} \\ &\quad \times\theta[\varepsilon]((2\omega_1)^{-1}u\,|\Omega). \end{aligned} \tag{73}$$

Applying ∂_{A_0} to (73) and set $u = 0$. Then, using (1) or (2) of Corollary 3, we have

$$\partial_{A_0}\theta[\tilde{\varepsilon}](0\,|\tilde{\Omega}) = \gamma\,(\det(C\Omega + D))^{1/2}\,\partial_{A_0}\theta[\varepsilon](0\,|\Omega). \tag{74}$$

On the other hand, as shown in [4], we have

$$\frac{1}{2}{}^t u\tilde{\eta}_1\tilde{\omega}_1^{-1}u = \frac{1}{2}{}^t u\eta_1\omega_1^{-1}u - \pi{}^t u^t(2\omega_1)^{-1}(C\Omega + D)^{-1}C(2\omega_1)^{-1}u. \tag{75}$$

By (73), (74), (75) we have

$$\exp(\frac{1}{2}{}^t u\tilde{\eta}_1\tilde{\omega}_1^{-1}u)\frac{\theta[\tilde{\varepsilon}]((2\tilde{\omega}_1)^{-1}u\,|\,\tilde{\Omega})}{\partial_{A_0}\theta[\tilde{\varepsilon}](0\,|\,\tilde{\Omega})}$$
$$= \exp(\frac{1}{2}{}^t u\eta_1\omega_1^{-1}u)\frac{\theta[\varepsilon]((2\omega_1)^{-1}u\,|\,\Omega)}{\partial_{A_0}\theta[\varepsilon](0\,|\,\Omega)},$$

which shows that $\sigma[\varepsilon](u)$ does not depend on the choice of canonical homology basis.
□

Acknowledgments

This research is supported by JSPS Grant-in-Aid for Scientific Research (C) No.23540245. A part of results of this paper was reported at the conference "Algebraic topology and abelian functions" held at Steklov Mathematical Institute, Moscow, in June 2013, at the conference "Geometry day" held at Sobolev Institute of Mathematics, Novosibirsk in August 2013 and at the conference "Algebraic curves with symmetries, their Jacobians and integrable dynamical systems" at National University of Kyiv-Mohyla Academy, Kiev in July 2014. I would like to thank organizers for invitations and kind hospitality at the conferences. I would also like to thank Harry Braden, Victor Buchstaber, Victor Enolski, Dmitry Korotkin, Dmitry Leykin, Andrey Mironov and Alexander Veselov for their interests in this work.

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